Improved Theory of Single-Particle Properties of Fermi Systems Using Sea-Bosons

Girish S. Setlur
Department of Physics and Materials Research Laboratory,
University of Illinois at Urbana-Champaign , Urbana Il 61801

In this article we combine the ideas introduced by us earlier in various proportions to arrive at a simple and yet powerful means of studying single-particle properties of homogeneous Fermi systems in detail without making assumptions regarding the validity or otherwise of Fermi-liquid theory. Novelties include the exact forms of the momentum distribution and spectral functions in the intermediate density regime and a dielectric function that is sensitive to significant qualitative changes in single-particle properties. This time, care is taken to include the particle-hole mode by interpreting the sum over modes as a weighted sum with the dynamical structure factor appearing as the weight.

I. INTRODUCTION

Both the Hartree-Fock[1] and the Random Phase approximations(RPA)[2] are widely used in the physics literature to study Fermi systems. The Bogoliubov theory[3] is the analog of the random-phase approximation for Bose systems. This latter fact has been demonstrated in detail in our article[4]. The analysis presented here could be repeated for Bose systems as well. But let us focus on the more important Fermi system. Here we try and address the question of validity of the Hartree-Fock approximation and the RPA. This is important since we showed in our earlier work that the simplest and most natural application of bosonization was in the regime where the RPA was exact[4]. It is therefore quite pertinent to ask whether this approximation is adequate. It is well-known that there are many problems with the RPA. In particular, the pair-correlation function has unphysical behaviour at short distances(it becomes negative)[6]. Therefore it is necessary to find better approximations. Many attempts have been made in the literature in this regard. The Hubbard approximation[?], the Singwi-Sjolander approach[?] and many others discussed in the text by Mahan[6] are among the more prominent. Many of them have found to resolve this problem of short distance behavior of the pair correlation function. However it seems that none of them are able to address the question of single-particle properties. This issue has acquired an urgency in the recent past given that a segment of the high T_c community is convinced that the non-superconducting state of these materials is non-Fermi liquid. Therefore being able to find a theory that describes the non-Fermi liquid regime accurately is desirable. In this attempt, we shall adopt a two-track approach. On the one hand we reinterpret the RPA so that it becomes more widely applicable. We argue that while the mean-field approximation carried out on the density operator is the Hartree-Fock approximation, the RPA manifests itself as mean-field theory applied not to the density operator but to the number operator. The density operator measures how the electrons are distributed in real space whereas the number operator measures how the electrons are distributed in momentum space. Just as the Hartree-Fock approximation is valid when fluctuations in the density of electrons at each point in real space is small, the RPA is valid when fluctuations in the momentum distribution of electrons is small compared with the average momentum distribution which measures the probability of an electron to possess a given momentum. We also introduce another mathematical transformation within the RPA-scheme and find that this maps a purely repulsive interaction to a purely attractive one but one that makes the particle exchange a momentum different from the usual one involved in two-body collisions. This unusual transformation allows us to define and compute yet another dielectric function of this system. A physical interpretation of this curious change in sign is given. A suitable combination of the purely repulsive(both actually and apparently) and purely attractive(only apparently, the electrons still repel one another) is shown to be better than either one spearately. In the end, we point out how a motivated researcher may be able to compute the phase diagram of the homogeneous electron gas by appropriately combining these three ingredients, although we ourselves restrict our attention to simple analytically solvable cases.

(i) Sea-bosons, indispensable for computing single-particle properties.

(ii) Generalised RPA and beyond (with fluctuations in the momentum distribution). Many new dielectric functions may be found here.

(iii) Repulsion-attraction duality : the enigmatic transformation that allows us to view the purely repulsive interaction between electrons as being attractive and repulsive at the same time when viewed in the sea-boson language.

In what follows, we try and combine the bosonization approach of our earlier articles with this notion of generalised RPA and repulsion-attraction duality and try and compute the single-particle properties. Just in order to whet the reader's appetite for the calculation of single-particle properties using sea-bosons, we pause here to explore another option, one that suggests itself quite naturally when one views RPA as a mean-field idea applied to the number operator rather than the density operator.

II. SINGLE-PARTICLE PROPERTIES WITHOUT USING BOSONIZATION

Let us try and write down some representative examples when RPA and Hartree-Fock approximations are used. Let us take for example, the jellium model[6].

$$H = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}} c_{\mathbf{k}} + \sum_{\mathbf{q}\neq 0} \frac{v_{\mathbf{q}}}{2V}(\rho_{\mathbf{q}}\rho_{-\mathbf{q}} - N) \tag{1}$$

If we apply the mean-field idea on the density, that is, replace $\rho_{\mathbf{q}}$ by $\langle \rho_{\mathbf{q}} \rangle$ then we get a hamiltonian that does not involve any coulomb interaction at all(apart from an additive constant).

$$H = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}} c_{\mathbf{k}} + \sum_{\mathbf{q}\neq 0} \frac{v_{\mathbf{q}}}{2V}(\langle\rho_{\mathbf{q}}\rangle\rho_{-\mathbf{q}} - N)$$

$$= \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}} c_{\mathbf{k}} - N\sum_{\mathbf{q}\neq 0} \frac{v_{\mathbf{q}}}{2V} \tag{2}$$

Therefore this approximation is bad in the extreme. However the random-phase approximation is still valid for this system. It is the mean-field idea applied not to the density but to the number operator. For this we have to rewrite the full hamiltonian given below,

$$H = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}} c_{\mathbf{k}} + \sum_{\mathbf{q}\neq 0} \frac{v_{\mathbf{q}}}{2V} \sum_{\mathbf{k},\mathbf{k}'} c^{\dagger}_{\mathbf{k}+\mathbf{q}/2} c^{\dagger}_{\mathbf{k}'-\mathbf{q}/2} c_{\mathbf{k}'+\mathbf{q}/2} c_{\mathbf{k}-\mathbf{q}/2} \tag{3}$$

This has to be replaced by,

$$H = H_0 - \sum_{\mathbf{q}\neq 0} \frac{v_{\mathbf{q}}}{2V} \sum_{\mathbf{k}\neq\mathbf{k}'} c^{\dagger}_{\mathbf{k}+\mathbf{q}/2} c_{\mathbf{k}'+\mathbf{q}/2} c^{\dagger}_{\mathbf{k}'-\mathbf{q}/2} c_{\mathbf{k}-\mathbf{q}/2} \tag{4}$$

where

$$H_0 = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}} c_{\mathbf{k}} - \sum_{\mathbf{q}\neq 0} \frac{v_{\mathbf{q}}}{2V} \sum_{\mathbf{k}} n_{\mathbf{k}+\mathbf{q}/2} n_{\mathbf{k}-\mathbf{q}/2} \tag{5}$$

where $n_{\mathbf{k}} = c^{\dagger}_{\mathbf{k}} c_{\mathbf{k}}$. Let us write,

$$n_{\mathbf{k}} = \langle n_{\mathbf{k}} \rangle + \delta n_{\mathbf{k}} \tag{6}$$

We plug the above decomposition into H_0 and find that if we neglect terms quadratic in the fluctuations, we get what we are after, namely the generalised RPA called for simplicity as just RPA.

$$H_{RPA} = E^{'}_0 + \sum_{\mathbf{k}} \tilde{\epsilon}_{\mathbf{k}} c^{\dagger}_{\mathbf{k}} c_{\mathbf{k}} \tag{7}$$

$$\tilde{\epsilon}_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \sum_{\mathbf{q}} \frac{v_{\mathbf{q}}}{V} \langle n_{\mathbf{k}-\mathbf{q}} \rangle \tag{8}$$

The average occupation is

$$\langle n_{\mathbf{k}} \rangle = \frac{1}{exp(\beta(\tilde{\epsilon}_{\mathbf{k}} - \mu)) + 1} \tag{9}$$

The chemical potential μ has to be fixed by making sure that,

$$\sum_{\mathbf{k}} \langle n_{\mathbf{k}} \rangle = \langle N \rangle \tag{10}$$

At zero temperature $\mu = \epsilon_F$, this quantity is equal to the usual Fermi energy when $v_{\mathbf{q}} = 0$.

$$\langle n_{\mathbf{k}} \rangle = \theta(\epsilon_F - \tilde{\epsilon}_{\mathbf{k}}) = \theta(\epsilon_F - \epsilon_{\mathbf{k}} + \sum_{\mathbf{q} \neq 0} \frac{v_{\mathbf{q}}}{V} \langle n_{\mathbf{k}-\mathbf{q}} \rangle) \tag{11}$$

and θ is the Heaviside step function. We can now demonstrate that the generalised RPA dielectric function(the Lindhard dielectric function[7] being a weak coupling limit of this) may be recovered using the following procedure. If one considers an extremely weak external perturbation applied to the system and follows the discussion in Mahan[6] one arrives at the following formula for the dielectric function,

$$\epsilon_{g-RPA}(\mathbf{q}, \omega) = 1 + \frac{v_{\mathbf{q}}}{V} \sum_{\mathbf{k}} \frac{\langle n_{\mathbf{k}+\mathbf{q}/2} \rangle - \langle n_{\mathbf{k}-\mathbf{q}/2} \rangle}{\omega - \tilde{\epsilon}_{\mathbf{k}+\mathbf{q}/2} + \tilde{\epsilon}_{\mathbf{k}-\mathbf{q}/2}} \tag{12}$$

The only point to bear in mind is that we have to use the full interacting momentum distribution rather than just its noninteracting value. When one includes fluctuations in the momentum distribution however, the answer given later, is very different from the usual RPA or even the generalised RPA. This feature of having the full momentum distribution in the numerator may be found in our earlier work[4]. Thus we can see that there is a whole new set of approximations that go beyond the RPA. While none of these revelations may come as a surprise to the reader, it should serve as a reminder that even our most cherished approximations may not be controlled in any sense of the term. It is more likely that they were the first to appear in the literature and probably the easiest ones to use thus explaining their popularity. The dielectric function written down above has the attractive feature of reducing to the familiar Lindhard dielectric function[7] for extremely weak coupling and at the same time giving us something very different for stronger coupling. Furthermore, if we ask the Luttinger liquid community to write down a dielectric function of the Luttinger liquid, they are in all probability, going to write down the traditional RPA dielectric function[?]. But we submit that this is a serious mistake. The more nonideal a system is, the more its dielectric function differs from the traditional RPA, even at high density. Parenthetically, we note that having granted the fact that the system is nonideal(namely, no Fermi surface or close to being devoid of one) it makes no difference whether the system is at high or low density, the point is, it is the *generalised* RPA that is still valid for this system. The only drawback of the above approach is that if we compute the one-particle Green functions we find that the imaginary part vanishes identically. This is unfortunate and we have to do better in order capture lifetime effects. The RPA hamiltonian neglects fluctuations in the momentum distribution of the electrons. In order to recover a finite lifetime of single particle excitations, we find that it is important to study the generalised H_0 rather than the more simple H_{RPA}. The fluctuations in the momentum distribution may be related to the mean by the following observation. Define,

$$N(\mathbf{k}, \mathbf{k}^{'}) = \langle n_{\mathbf{k}} n_{\mathbf{k}'} \rangle - \langle n_{\mathbf{k}} \rangle \langle n_{\mathbf{k}'} \rangle \tag{13}$$

. The fluctuation in the number operator is $N(\mathbf{k}, \mathbf{k}) = \langle n_{\mathbf{k}}^2 \rangle - \langle n_{\mathbf{k}} \rangle^2$. Since $n_{\mathbf{k}}^2 = n_{\mathbf{k}}$ for fermions, we have

$$N(\mathbf{k}, \mathbf{k}) = \langle n_{\mathbf{k}} \rangle (1 - \langle n_{\mathbf{k}} \rangle) \tag{14}$$

Therefore, we may conclude that any nonideal momentum distribution fluctuates (there are however, pathological exceptions, see footnote[12]). In fact, a very nonideal momentum distribution such as one for which $\langle n_{\mathbf{k}} \rangle = 0.5$ for most momenta has the largest fluctuation. When dealing with nonideal systems, we are

obliged to consider fluctuations in the momentum distribution. In order to study lifetime effects therefore, we have to include the full H_0. This may be written more transparently as (apart from additive constants) $H_0 = H_{RPA} + H_{fl}$,

$$H_{fl} = -\sum_{\mathbf{q}\neq 0}\frac{v_{\mathbf{q}}}{2V}\sum_{\mathbf{k}}\delta n_{\mathbf{k}+\mathbf{q}/2}\delta n_{\mathbf{k}-\mathbf{q}/2} \tag{15}$$

The full Fermi propagator may be evaluated by treating the fluctuation part as a perturbation and using the functional methods of Schwinger illustrated brilliantly by Kadanoff and Baym[8]. Before we plunge into the details it is important to keep in mind that the mean also changes when we consider fluctuations. That is,

$$\langle n_{\mathbf{k}}\rangle = \langle n_{\mathbf{k}}\rangle_{RPA} + \langle n_{\mathbf{k}}\rangle_{fl} \tag{16}$$

here $\langle n_{\mathbf{k}}\rangle_{RPA}$ is given by Eq.(9) or Eq.(11) the rest is nonzero only when the fluctuation in the momentum distribution is large (which is, unfortunately, almost always the case when $\langle n_{\mathbf{k}}\rangle_{RPA}$ is nonideal) Therefore, now, $\delta n_{\mathbf{k}}$ refers to fluctuation around the full average rather than the RPA average. The final answers are given below, and it is hoped that the reader can rederive them using the references quoted in the bibliography (mainly[6],[8]). We shall adhere to the notation of Kadanoff and Baym[8]. In their notation the final answers for the single-particle Green functions are as follows(we assume in the following that $F(\mathbf{p}) \neq 0$, however one may investigate the limit $F(\mathbf{p}) \to 0$, here $z_n = (2n+1)\pi/\beta$),

$$G_n(\mathbf{k}) = \frac{1}{iz_n - \tilde{\epsilon}_{\mathbf{k}} + \mu - \Sigma_n(\mathbf{k})} \tag{17}$$

and

$$\Sigma_n(\mathbf{k}) = G_n(\mathbf{k})F(\mathbf{k}) \tag{18}$$

$$F(\mathbf{k}) = \sum_{\mathbf{q},\mathbf{q}'\neq 0}\frac{v_{\mathbf{q}}v_{\mathbf{q}'}}{V^2}N(\mathbf{k}-\mathbf{q},\mathbf{k}-\mathbf{q}^{'}) \tag{19}$$

From Eq.(18) we may obtain the real and imaginary parts of the retarded self-energy, and from there the spectral function and the collision rates[8].

$$\Gamma(\mathbf{p},\omega) = \sqrt{-\kappa(\mathbf{p},\omega)} \tag{20}$$

Similarly,

$$A(\mathbf{p},\omega) = \frac{\sqrt{-\kappa(\mathbf{p},\omega)}}{F(\mathbf{p})} \tag{21}$$

if $\kappa(\mathbf{p},\omega) = (\omega - \tilde{\epsilon}_{\mathbf{p}} + \mu)^2 - 4F(\mathbf{p}) < 0$ and both are zero otherwise(when $F(\mathbf{p}) \neq 0$). It can be seen that the spectral function is peaked around $\tilde{\epsilon}_{\mathbf{p}} - \mu$ with a width of the order of $2\sqrt{F(\mathbf{p})}$, and the collision rate is vanishingly small for those values of $\mathbf{p}$ for which $F(\mathbf{p})$ is close to zero. It may be shown that the momentum distribution including possible fluctuations is given by,

$$\langle n_{\mathbf{p}}\rangle = (\frac{2}{\pi})\int_{-\pi/2}^{\pi/2} d\theta\ cos^2\theta\ \frac{1}{e^{\beta(\tilde{\epsilon}_{\mathbf{p}}-\mu)}e^{2\beta\sqrt{F(\mathbf{p})}sin\theta} + 1} \tag{22}$$

and $\langle n_{\mathbf{p}}\rangle = 0$ for $F(\mathbf{p}) < 0$. Again, it may be seen quite easily that Eq.(22) is identical to Eq.(9) when fluctuations in the momentum distribution are ignored. The only sticking point now is the computation of the fluctuation in the momentum distribution. This may be done in a similar manner by employing the functional methods of Kadanoff and Baym[8]. However, we shall defer the computation of this quantity until the next section. Incidentally, the formulas we suggested in our preprints for this quantity in retrospect, are probably rather poor. Instead we shall adopt a more rigorous approach in the next section. The dielectric function is also modified as a result of these fluctuations and the final formula for the dielectric function including possible fluctuations in the momentum distribution is,

$$\epsilon_{eff}(\mathbf{q},\omega) = \epsilon_{g-RPA}(\mathbf{q},\omega) - (\frac{v_{\mathbf{q}}}{V})^2 \frac{P_2(\mathbf{q},\omega)}{\epsilon_{g-RPA}(\mathbf{q},\omega)} \tag{23}$$

Here,

$$P_2(\mathbf{q},\omega) = \sum_{\mathbf{k},\mathbf{k}'} \frac{N(\mathbf{k}+\mathbf{q}/2,\mathbf{k}^{'}+\mathbf{q}/2) - N(\mathbf{k}-\mathbf{q}/2,\mathbf{k}^{'}+\mathbf{q}/2) - N(\mathbf{k}+\mathbf{q}/2,\mathbf{k}^{'}-\mathbf{q}/2) + N(\mathbf{k}-\mathbf{q}/2,\mathbf{k}^{'}-\mathbf{q}/2)}{(\omega - \tilde{\epsilon}_{\mathbf{k}-\mathbf{q}/2} + \tilde{\epsilon}_{\mathbf{k}+\mathbf{q}/2})(\omega - \tilde{\epsilon}_{\mathbf{k}'-\mathbf{q}/2} + \tilde{\epsilon}_{\mathbf{k}'+\mathbf{q}/2})} \tag{24}$$

A full derivation of this is relegated to the appendix in order to avoid interupting the flow of ideas. Assuming that ω has a small imaginary part, we recover both the real and imaginary parts of the full dielectric function.

III. REPULSION-ATTRACTION DUALITY

In this section, we point out an interesting and even a seemingly paradoxical duality between attraction and repulsion. In a homogeneous electron gas, electron repel each other. However, we may show that by virtue of the electron being fermions, an exchange operation on the interaction part of the full hamiltonian leads to a change in sign of the interaction and also a change in the momentum exchanged by the interacting electrons. To see this, let us write down the interaction part of the full hamiltonian again.

$$H_{int} = \sum_{\mathbf{q}\neq 0} \frac{v_{\mathbf{q}}}{2V} \sum_{\mathbf{k}\neq\mathbf{k}'} c^{\dagger}_{\mathbf{k}+\mathbf{q}/2} c^{\dagger}_{\mathbf{k}'-\mathbf{q}/2} c_{\mathbf{k}'+\mathbf{q}/2} c_{\mathbf{k}-\mathbf{q}/2} \tag{25}$$

We may rearrange the various fermion operators in this interaction so that precisely the same quantity may be rewritten as,

$$H_{int} = -\sum_{\mathbf{k}\neq\mathbf{k}'} \frac{v_{\mathbf{k}-\mathbf{k}'}}{2V} \sum_{\mathbf{q}\neq 0} c^{\dagger}_{\mathbf{k}+\mathbf{q}/2} c^{\dagger}_{\mathbf{k}'-\mathbf{q}/2} c_{\mathbf{k}'+\mathbf{q}/2} c_{\mathbf{k}-\mathbf{q}/2} \tag{26}$$

Two important changes have occured. First there is the change in sign. This is the reason for the term repulsion-attraction duality. The second is the momentum that is exchanged is different. The momentum carried away by the virtual photon is no longer $\mathbf{q}$ but is $\mathbf{k}-\mathbf{k}^{'}$. The change in sign is peculiar to fermions, whereas the other feature is present for bosons too. Let us now use this new form and recompute the dielectric function. Actually, we could have done the same shuffle for the $\mathbf{k}=\mathbf{k}^{'}$ part of the interaction. But we shall relegate a more careful examination of these issues to future publications. The main purpose of the present article is to lay before the reader a scheme that is robust, rich and sufficiently general so that she may apply the ideas to other more practical problems. In order to compute the dielectric function suggested by this shuffled interaction let us proceed as follows. The full hamiltonian may be written as ($n_0(\mathbf{k}) = n_{\mathbf{k}} = c^{\dagger}_{\mathbf{k}} c_{\mathbf{k}}$),

$$H = \sum_{\mathbf{k}} \tilde{\epsilon}_{\mathbf{k}} n_0(\mathbf{k}) - \sum_{\mathbf{q}\neq 0} \frac{v(\mathbf{q})}{2V} \sum_{\mathbf{k}} \delta n_0(\mathbf{k}+\mathbf{q}/2)\ \delta n_0(\mathbf{k}-\mathbf{q}/2)$$

$$- \sum_{\mathbf{k}\neq\mathbf{k}'} \frac{v_{\mathbf{k}-\mathbf{k}'}}{2V} \sum_{\mathbf{q}\neq 0} n_{\mathbf{q}}(\mathbf{k}) n_{-\mathbf{q}}(\mathbf{k}^{'}) \tag{27}$$

here $n_{\mathbf{q}}(\mathbf{k}) = c^{\dagger}_{\mathbf{k}+\mathbf{q}/2} c_{\mathbf{k}-\mathbf{q}/2}$ Let us now apply an external field,

$$H_{ext}(t) = \sum_{\mathbf{k},\mathbf{q}} (U_{ext}(\mathbf{q},t) + U^{*}_{ext}(-\mathbf{q},t)) n_{\mathbf{q}}(\mathbf{k}) \tag{28}$$

In order to proceed, we appeal to the random-phase approximation. In a suitably generalised sense, it may be defined to mean the following approximation :

$$[n_{\mathbf{q}}(\mathbf{k}), n_{\mathbf{q}'}(\mathbf{k}^{'})] = c^{\dagger}_{\mathbf{k}+\mathbf{q}/2} c_{\mathbf{k}'-\mathbf{q}'/2} \delta_{\mathbf{k}-\mathbf{q}/2,\mathbf{k}'+\mathbf{q}'/2} - c^{\dagger}_{\mathbf{k}'+\mathbf{q}'/2} c_{\mathbf{k}-\mathbf{q}/2} \delta_{\mathbf{k}'-\mathbf{q}'/2,\mathbf{k}+\mathbf{q}/2}$$

$$\approx \delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},-\mathbf{q}'}(n_0(\mathbf{k}+\mathbf{q}/2)-n_0(\mathbf{k}-\mathbf{q}/2)) \tag{29}$$

$$[n_{\mathbf{q}}(\mathbf{k}), n_0(\mathbf{p})] = n_{\mathbf{q}}(\mathbf{k})(\delta_{\mathbf{p},\mathbf{k}-\mathbf{q}/2}-\delta_{\mathbf{p},\mathbf{k}+\mathbf{q}/2}) \tag{30}$$

Here we have to retain $n_0(\mathbf{k})$ as an operator in order to include effects due to fluctuations in the momentum distribution. Let us now write down the equation of motion of this operator.

$$i\frac{\partial}{\partial t}n_{\mathbf{q}}(\mathbf{k}) = (\tilde{\epsilon}_{\mathbf{k}-\mathbf{q}/2}-\tilde{\epsilon}_{\mathbf{k}+\mathbf{q}/2})n_{\mathbf{q}}(\mathbf{k}) - \sum_{\mathbf{q}'\neq 0}\frac{v(\mathbf{q}')}{V}(\delta n_0(\mathbf{k}-\mathbf{q}/2-\mathbf{q}')n_{\mathbf{q}}(\mathbf{k})$$

$$-\delta n_0(\mathbf{k}+\mathbf{q}/2-\mathbf{q}')n_{\mathbf{q}}(\mathbf{k})) - \sum_{\mathbf{k}\neq\mathbf{k}'}\frac{v_{\mathbf{k}-\mathbf{k}'}}{V}(n_0(\mathbf{k}+\mathbf{q}/2)n_{\mathbf{q}}(\mathbf{k}')-n_0(\mathbf{k}-\mathbf{q}/2)n_{\mathbf{q}}(\mathbf{k}'))$$

$$+(U_{ext}(-\mathbf{q},t)+U^{*}_{ext}(\mathbf{q},t))(n_0(\mathbf{k}+\mathbf{q}/2)-n_0(\mathbf{k}-\mathbf{q}/2)) \tag{31}$$

Write $n_0(\mathbf{k}) = \langle n_0(\mathbf{k})\rangle + \delta\ n_0(\mathbf{k})$

$$i\frac{\partial}{\partial t}\langle n_{\mathbf{q}}(\mathbf{k})\rangle = (\tilde{\epsilon}_{\mathbf{k}-\mathbf{q}/2}-\tilde{\epsilon}_{\mathbf{k}+\mathbf{q}/2})\langle n_{\mathbf{q}}(\mathbf{k})\rangle$$

$$-\sum_{\mathbf{q}'\neq 0}\frac{v(\mathbf{q}')}{V}(\langle\delta n_0(\mathbf{k}-\mathbf{q}/2-\mathbf{q}')n_{\mathbf{q}}(\mathbf{k})\rangle-\langle\delta n_0(\mathbf{k}+\mathbf{q}/2-\mathbf{q}')n_{\mathbf{q}}(\mathbf{k})\rangle)$$

$$-\sum_{\mathbf{k}\neq\mathbf{k}'}\frac{v_{\mathbf{k}-\mathbf{k}'}}{V}(\langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle-\langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle)\langle n_{\mathbf{q}}(\mathbf{k}')\rangle$$

$$-\sum_{\mathbf{k}\neq\mathbf{k}'}\frac{v_{\mathbf{k}-\mathbf{k}'}}{V}(\langle\delta n_0(\mathbf{k}+\mathbf{q}/2)n_{\mathbf{q}}(\mathbf{k}')\rangle-\langle\delta n_0(\mathbf{k}-\mathbf{q}/2)n_{\mathbf{q}}(\mathbf{k}')\rangle)$$

$$+(U_{ext}(-\mathbf{q},t)+U^{*}_{ext}(\mathbf{q},t))(\langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle-\langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle) \tag{32}$$

$$i\frac{\partial}{\partial t}\langle\delta n_0(\mathbf{p})n_{\mathbf{q}}(\mathbf{k})\rangle = (\tilde{\epsilon}_{\mathbf{k}-\mathbf{q}/2}-\tilde{\epsilon}_{\mathbf{k}+\mathbf{q}/2})\langle\delta n_0(\mathbf{p})n_{\mathbf{q}}(\mathbf{k})\rangle$$

$$-\sum_{\mathbf{q}'\neq 0}\frac{v(\mathbf{q}')}{V}(\langle\delta n_0(\mathbf{p})\delta n_0(\mathbf{k}-\mathbf{q}/2-\mathbf{q}')\rangle-\langle\delta n_0(\mathbf{p})\delta n_0(\mathbf{k}+\mathbf{q}/2-\mathbf{q}')\rangle)\langle n_{\mathbf{q}}(\mathbf{k})\rangle$$

$$-\sum_{\mathbf{k}\neq\mathbf{k}'}\frac{v_{\mathbf{k}-\mathbf{k}'}}{V}(\langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle-\langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle)\langle\delta n_0(\mathbf{p})n_{\mathbf{q}}(\mathbf{k}')\rangle$$

$$-\sum_{\mathbf{k}\neq\mathbf{k}'}\frac{v_{\mathbf{k}-\mathbf{k}'}}{V}(\langle\delta n_0(\mathbf{p})\delta n_0(\mathbf{k}+\mathbf{q}/2)\rangle-\langle\delta n_0(\mathbf{p})\delta n_0(\mathbf{k}-\mathbf{q}/2)\rangle)\langle n_{\mathbf{q}}(\mathbf{k}')\rangle$$

$$+(U_{ext}(-\mathbf{q},t)+U^{*}_{ext}(\mathbf{q},t))(\langle\delta n_0(\mathbf{p})\delta n_0(\mathbf{k}+\mathbf{q}/2)\rangle-\langle\delta n_0(\mathbf{p})\delta n_0(\mathbf{k}-\mathbf{q}/2)\rangle) \tag{33}$$

Assuming that,

$$U_{ext}(-\mathbf{q},t) = U_{ext}(-\mathbf{q},0)e^{-i\omega\ t} \tag{34}$$

$$(\omega - \tilde{\epsilon}_{\mathbf{k-q}/2} + \tilde{\epsilon}_{\mathbf{k+q}/2})\langle\ \delta\ n_0(\mathbf{p})\ n_{\mathbf{q}}(\mathbf{k})\rangle = -\sum_{\mathbf{q}^{'}\neq 0}\frac{v_{\mathbf{q}^{'}}}{V}(N(\mathbf{p},\mathbf{k}-\mathbf{q}/2-\mathbf{q}^{'}) - N(\mathbf{p},\mathbf{k}+\mathbf{q}/2-\mathbf{q}^{'}))\langle n_{\mathbf{q}}(\mathbf{k})\rangle$$

$$-(\langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle - \langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle)\sum_{\mathbf{k}^{'}\neq\mathbf{k}}\frac{v_{\mathbf{k}-\mathbf{k}^{'}}}{V}\langle\delta\ n_0(\mathbf{p})n_{\mathbf{q}}(\mathbf{k}^{'})\rangle$$

$$-(N(\mathbf{p},\mathbf{k}+\mathbf{q}/2) - N(\mathbf{p},\mathbf{k}-\mathbf{q}/2))\sum_{\mathbf{k}^{'}\neq\mathbf{k}}\frac{v_{\mathbf{k}-\mathbf{k}^{'}}}{V}\langle n_{\mathbf{q}}(\mathbf{k}^{'})\rangle$$

$$+U_{ext}(-\mathbf{q},0)(N(\mathbf{p},\mathbf{k}+\mathbf{q}/2) - N(\mathbf{p},\mathbf{k}-\mathbf{q}/2)) \tag{35}$$

$$(\omega - \tilde{\epsilon}_{\mathbf{k-q}/2} + \tilde{\epsilon}_{\mathbf{k+q}/2})\langle n_{\mathbf{q}}(\mathbf{k})\rangle = -\sum_{\mathbf{q}^{'}\neq 0}\frac{v_{\mathbf{q}^{'}}}{V}(\langle\delta n_0(\mathbf{k}-\mathbf{q}/2-\mathbf{q}^{'})n_{\mathbf{q}}(\mathbf{k})\rangle - \langle\delta n_0(\mathbf{k}+\mathbf{q}/2-\mathbf{q}^{'})n_{\mathbf{q}}(\mathbf{k})\rangle)$$

$$-(\langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle - \langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle)\sum_{\mathbf{k}^{'}\neq\mathbf{k}}\frac{v_{\mathbf{k}-\mathbf{k}^{'}}}{V}\langle n_{\mathbf{q}}(\mathbf{k}^{'})\rangle$$

$$-\sum_{\mathbf{k}^{'}\neq\mathbf{k}}\frac{v_{\mathbf{k}-\mathbf{k}^{'}}}{V}(\langle\delta n_0(\mathbf{k}+\mathbf{q}/2)n_{\mathbf{q}}(\mathbf{k}^{'})\rangle - \langle\delta n_0(\mathbf{k}-\mathbf{q}/2)n_{\mathbf{q}}(\mathbf{k}^{'})\rangle)$$

$$+U_{ext}(-\mathbf{q},0)(\langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle - \langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle) \tag{36}$$

In order to simplify this further, let us assume that we are in the weakly nonideal regime. That is, it is legitimate to treat the momentum distribution as possessing a sharp Fermi surface and no other striking features. Let us now define,

$$\tilde{n}_{\mathbf{q}}(\vec{r}) = \frac{1}{V}\sum_{\mathbf{k}}e^{-i\mathbf{k}.\vec{r}}n_{\mathbf{q}}(\mathbf{k}) \tag{37}$$

$$(\omega+i\frac{\mathbf{q}.\nabla}{m})\langle\ \delta\ n_0(\mathbf{p})\ \tilde{n}_{\mathbf{q}}(\vec{r})\rangle = -\frac{1}{V}\sum_{\mathbf{k},\mathbf{q}^{'}\neq 0}\frac{v_{\mathbf{q}^{'}}}{V}\int d^3r^{'}\ v(r^{'})\ e^{i\mathbf{k}.(\vec{r}^{'}-\vec{r})}(N(\mathbf{p},\mathbf{k}-\mathbf{q}/2-\mathbf{q}^{'})-N(\mathbf{p},\mathbf{k}+\mathbf{q}/2-\mathbf{q}^{'}))\ \langle\tilde{n}_{\mathbf{q}}(\vec{r}^{'})\rangle$$

$$-\frac{1}{V}\int d^3r^{'}\ \sum_{\mathbf{k}}e^{i\mathbf{k}.(\vec{r}^{'}-\vec{r})}(\langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle - \langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle)v(r^{'})\langle\delta\ n_0(\mathbf{p})n_{\mathbf{q}}(\vec{r}^{'})\rangle$$

$$-\frac{1}{V}\int d^3r^{'}\ \sum_{\mathbf{k}}e^{i\mathbf{k}.(\vec{r}^{'}-\vec{r})}(N(\mathbf{p},\mathbf{k}+\mathbf{q}/2) - N(\mathbf{p},\mathbf{k}-\mathbf{q}/2))v(r^{'})\langle n_{\mathbf{q}}(\vec{r}^{'})\rangle$$

$$+U_{ext}(-\mathbf{q},0)\frac{1}{V}\sum_{\mathbf{k}}e^{-i\mathbf{k}.\vec{r}}(N(\mathbf{p},\mathbf{k}+\mathbf{q}/2) - N(\mathbf{p},\mathbf{k}-\mathbf{q}/2)) \tag{38}$$

$$(\omega + i\frac{\mathbf{q}.\nabla}{m})\langle \tilde{n}_{\mathbf{q}}(\vec{r})\rangle = -\frac{1}{V}\sum_{\mathbf{k},\mathbf{q}^{'}\neq 0}\frac{v_{\mathbf{q}'}}{V}e^{i\mathbf{k}.(\vec{r}^{'}-\vec{r})}(\langle \delta n_0(\mathbf{k}-\mathbf{q}/2-\mathbf{q}^{'})n_{\mathbf{q}}(\mathbf{k})\rangle - \langle \delta n_0(\mathbf{k}+\mathbf{q}/2-\mathbf{q}^{'})n_{\mathbf{q}}(\vec{r}^{'})\rangle)$$

$$-\frac{1}{V}\sum_{\mathbf{k}}\int d^3r^{'}\ v(r^{'})e^{i\mathbf{k}.(\vec{r}^{'}-\vec{r})}(\langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle - \langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle)\langle n_{\mathbf{q}}(\vec{r}^{'})\rangle$$

$$-\frac{1}{V}\sum_{\mathbf{k}}e^{i\mathbf{k}.(\vec{r}^{'}-\vec{r})}\int d^3r^{'}\ v(r^{'})(\langle \delta n_0(\mathbf{k}+\mathbf{q}/2)n_{\mathbf{q}}(\vec{r}^{'})\rangle - \langle \delta n_0(\mathbf{k}-\mathbf{q}/2)n_{\mathbf{q}}(\vec{r}^{'})\rangle)$$

$$+U_{ext}(-\mathbf{q},0)\frac{1}{V}\sum_{\mathbf{k}}e^{-i\mathbf{k}.\vec{r}}(\langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle - \langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle) \tag{39}$$

Let us now compute the following quantity,

$$f_{\mathbf{q}}(\vec{R}) = \frac{1}{V}\sum_{\mathbf{k}}e^{i\mathbf{k}.(\vec{r}^{'}-\vec{r})}(\langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle - \langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle) \tag{40}$$

where $\vec{R} = \vec{r}^{'} - \vec{r}$. If k_f is sufficiently large ($k_f >> q, k_f >> 1/R$) we may write,

$$f_{\mathbf{q}}(\vec{R}) \approx C_0\mathbf{q}.\nabla_R\delta^3(R) \tag{41}$$

$$C_0 = \frac{(4\pi)^2}{(2\pi)^3}(\frac{i}{q})\int_0^{\infty}dR\ \frac{2^3}{q^2R^2}(sin(k_fR) - (k_fR)cos(k_fR))(sin(\frac{qR}{2}) - (\frac{qR}{2})cos(\frac{qR}{2})) \tag{42}$$

$$(\omega+i\frac{\mathbf{q}.\nabla}{m})\langle \delta n_0(\mathbf{p})\ \tilde{n}_{\mathbf{q}}(\vec{r})\rangle = -\frac{1}{V}\sum_{\mathbf{k},\mathbf{q}^{'}\neq 0}\frac{v_{\mathbf{q}'}}{V}\int d^3r^{'}\ v(r^{'})\ e^{i\mathbf{k}.(\vec{r}^{'}-\vec{r})}(N(\mathbf{p},\mathbf{k}-\mathbf{q}/2-\mathbf{q}^{'})-N(\mathbf{p},\mathbf{k}+\mathbf{q}/2-\mathbf{q}^{'}))\ \langle \tilde{n}_{\mathbf{q}}(\vec{r}^{'})\rangle$$

$$+C_0\mathbf{q}.\nabla_{\vec{r}}v(r)\langle \delta n_0(\mathbf{p})\tilde{n}_{\mathbf{q}}(\vec{r})\rangle$$

$$-\frac{1}{V}\int d^3r^{'}\ \sum_{\mathbf{k}}e^{i\mathbf{k}.(\vec{r}^{'}-\vec{r})}(N(\mathbf{p},\mathbf{k}+\mathbf{q}/2) - N(\mathbf{p},\mathbf{k}-\mathbf{q}/2))v(r^{'})\langle n_{\mathbf{q}}(\vec{r}^{'})\rangle$$

$$+U_{ext}(-\mathbf{q},0)\frac{1}{V}\sum_{\mathbf{k}}e^{-i\mathbf{k}.\vec{r}}(N(\mathbf{p},\mathbf{k}+\mathbf{q}/2) - N(\mathbf{p},\mathbf{k}-\mathbf{q}/2)) \tag{43}$$

$$(\omega+i\frac{\mathbf{q}.\nabla}{m})\langle \tilde{n}_{\mathbf{q}}(\vec{r})\rangle = -\frac{1}{V}\sum_{\mathbf{k},\mathbf{q}^{'}\neq 0}\frac{v_{\mathbf{q}'}}{V}\int d^3r^{'}\ v(r^{'})e^{i\mathbf{k}.(\vec{r}^{'}-\vec{r})}(\langle \delta n_0(\mathbf{k}-\mathbf{q}/2-\mathbf{q}^{'})n_{\mathbf{q}}(\vec{r}^{'})\rangle-\langle \delta n_0(\mathbf{k}+\mathbf{q}/2-\mathbf{q}^{'})n_{\mathbf{q}}(\vec{r}^{'})\rangle)$$

$$+C_0\mathbf{q}.\nabla_{\vec{r}}v(r)\langle \tilde{n}_{\mathbf{q}}(\vec{r})\rangle$$

$$-\frac{1}{V}\sum_{\mathbf{k}}e^{i\mathbf{k}.(\vec{r}^{'}-\vec{r})}\int d^3r^{'}\ v(r^{'})(\langle \delta n_0(\mathbf{k}+\mathbf{q}/2)n_{\mathbf{q}}(\vec{r}^{'})\rangle - \langle \delta n_0(\mathbf{k}-\mathbf{q}/2)n_{\mathbf{q}}(\vec{r}^{'})\rangle)$$

$$+U_{ext}(-\mathbf{q},0)\frac{1}{V}\sum_{\mathbf{k}}e^{-i\mathbf{k}.\vec{r}}(\langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle - \langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle) \tag{44}$$

At this stage the author is unable to complete this calculation, but hopefully the reader appreciates the spirit of the discussion. The author apologises in advance for this omission.

IV. IMPROVED THEORY OF SINGLE-PARTICLE PROPERTIES USING BOSONIZATION

In this section, we use the sea-boson method suitably generalised to accomodate fluctuations in the momentum distribution to compute single-particle properties. In the appendix we show how to derive the dielectric function using this method. The dielectric function may be evaluated by more conventional means as well[11]. However, the single-particle properties evaluated using the sea-boson method is more accurate as one is able to treat the problem more systematically and the physical interpretation of the formulas is also more transparent. The really systematic approach would be to write down the sea-boson correspondence suitably generalised to accomodate fluctuations in the momentum distribution and evaluate the various propagators like we did in our earlier work[4]. However, we shall adopt a simpler but hopefully the correct approach here. The idea is to simply borrow from our earlier work except that the zeros of the dielectric function are no longer the zeros of the RPA dielectric function but that of ϵ_{eff} and we need to interpret the derivate of the dielectric funcion with respect to frequency that appears in these formulas as being the derivative of the full dielectric function. Therefore, let us write down the various formulas. The full hamiltonian in diagonalised form has the following appearence,

$$H_{full} = \sum_{\mathbf{q},i} \omega_i(\mathbf{q}) d_i^{\dagger}(\mathbf{q}) d_i(\mathbf{q}) \tag{45}$$

$$\bar{n}_{\mathbf{k}} = n^{\beta}(\mathbf{k})\ F_1(\mathbf{k}) + (1 - n^{\beta}(\mathbf{k}))\ F_2(\mathbf{k}) \tag{46}$$

where,

$$F_1(\mathbf{k}) = \frac{1}{1 + \frac{S_B(\mathbf{k})}{1+S_A(\mathbf{k})}} \tag{47}$$

$$F_2(\mathbf{k}) = \frac{1}{1 + \frac{1+S_B(\mathbf{k})}{S_A(\mathbf{k})}} \tag{48}$$

$$S_A(\mathbf{k}) = \sum_{\mathbf{q},i} \frac{\bar{n}_{\mathbf{k-q}}}{(\omega_i(-\mathbf{q}) + \mathbf{k}.\mathbf{q}/m - \epsilon_{\mathbf{q}})^2} g_i^2(-\mathbf{q}) \tag{49}$$

$$S_B(\mathbf{k}) = \sum_{\mathbf{q},i} \frac{1 - \bar{n}_{\mathbf{k+q}}}{(\omega_i(-\mathbf{q}) + \mathbf{k}.\mathbf{q}/m + \epsilon_{\mathbf{q}})^2} g_i^2(-\mathbf{q}) \tag{50}$$

$$g_i(\mathbf{q}) = [\sum_{\mathbf{k}} \frac{\bar{n}_{\mathbf{k-q}/2} - \bar{n}_{\mathbf{k+q}/2}}{(\omega_i(\mathbf{q}) - \frac{\mathbf{k}.\mathbf{q}}{m})^2}]^{-\frac{1}{2}} \tag{51}$$

It may be observed that this quantity is just the frequency derivative of the polarization. In other words,

$$g_i^{-2}(\mathbf{q}) = \frac{V}{v_{\mathbf{q}}}(\frac{\partial}{\partial\omega})_{\omega=\omega_i(\mathbf{q})} \epsilon^{P}_{g-RPA}(\mathbf{q},\omega) \tag{52}$$

By analogy we may generalise this so that when fluctuations are introduced,

$$g_i^{-2}(\mathbf{q}) = \frac{V}{v_{\mathbf{q}}}(\frac{\partial}{\partial\omega})_{\omega=\omega_i(\mathbf{q})} \epsilon^{P}_{eff}(\mathbf{q},\omega) \tag{53}$$

We shall employ this latter definition in our analysis. $\omega_i(\mathbf{q})$ is the zero of the overall dielectric function,

$$\epsilon^{P}_{eff}(\mathbf{q},\omega_i) = 0 \tag{54}$$

It is worthwhile simplifying the effective dielectric function.

$$\epsilon_{eff}(\mathbf{q},\omega) = \epsilon_{g-RPA}(\mathbf{q},\omega) + \frac{(\epsilon_{g-RPA}(\mathbf{q},\omega) - 1)(\epsilon_{g-RPA}(\mathbf{q},\omega) - \epsilon_{\beta}(\mathbf{q},\omega))}{\epsilon_{g-RPA}(\mathbf{q},\omega)} \tag{55}$$

The principal part of this is given by,

$$\epsilon^P_{eff}(\mathbf{q},\omega) = \epsilon^P_{g-RPA}(\mathbf{q},\omega) + \frac{(\epsilon^P_{g-RPA}(\mathbf{q},\omega) - 1)(\epsilon^P_{g-RPA}(\mathbf{q},\omega) - \epsilon^P_{\beta}(\mathbf{q},\omega))}{\epsilon^P_{g-RPA}(\mathbf{q},\omega)} \tag{56}$$

where,

$$\epsilon_{g-RPA}(\mathbf{q},\omega) = 1 + \frac{v_{\mathbf{q}}}{V}\sum_{\mathbf{k}} \frac{\bar{n}_{\mathbf{k}+\mathbf{q}/2} - \bar{n}_{\mathbf{k}-\mathbf{q}/2}}{\omega - \frac{\mathbf{k}.\mathbf{q}}{m}} \tag{57}$$

$$\epsilon_{\beta}(\mathbf{q},\omega) = 1 + \frac{v_{\mathbf{q}}}{V}\sum_{\mathbf{k}} \frac{n^{\beta}_{\mathbf{k}+\mathbf{q}/2} - n^{\beta}_{\mathbf{k}-\mathbf{q}/2}}{\omega - \frac{\mathbf{k}.\mathbf{q}}{m}} \tag{58}$$

and the principal parts of these functions are just the real parts.

$$\epsilon^P_{g-RPA}(\mathbf{q},\omega) = 1 + \frac{v_{\mathbf{q}}}{V}\sum_{\mathbf{k}} \; P \; \frac{\bar{n}_{\mathbf{k}+\mathbf{q}/2} - \bar{n}_{\mathbf{k}-\mathbf{q}/2}}{\omega - \frac{\mathbf{k}.\mathbf{q}}{m}} \tag{59}$$

A. Just How Many Zeros Does the Dielectric Function Really Have ?

In this section, we address what is perhaps the most vexing problem in this approach. What is the size of the set to which i in Eq.(45) belongs ? If one counts only the collective mode which is what a naive approach would lead us to do, then we are ignoring a large (infinite !) number of particle-hole modes. And yet, it seems that the particle-hole modes don't come about naturally and must be forced into the formalism[4]. It is really important to address this issue since ignoring it would mean that the excitation spectrum of the homogeneous electron gas possesses a gap when in fact it should not. In other words, if one counts only the collective mode(plasmon) one arrives at the unavoidable conclusion that the excitation spectrum has a finite gap(equal to the plasmon energy). How then should one introduce the particle-hole mode ? In this section, we show how to introduce both the particle-hole mode and collective mode in a unified manner. In our previous work[4] we suggested that the zeros of the RPA dielectric function should be interpreted as the maxima of the dynamical structure factor. This is quite an unusual and drastic departure from the notion of a root of a function. Being physicists we accept it as it has a physical interpretation. Let us examine the diagonalised form of the full hamiltonian[4]:

$$H_{full} = \sum_{\mathbf{q},i} \omega_i(\mathbf{q}) d^{\dagger}_i(\mathbf{q}) d_i(\mathbf{q}) \tag{60}$$

For small $\mathbf{q}$, we expect this sum to involve just the collective mode. In other words,

$$H_{full} = \sum_{\mathbf{q}=small} \omega_c(\mathbf{q}) d^{\dagger}_c(\mathbf{q}) d_c(\mathbf{q}) \tag{61}$$

This may be achieved by employing the following device,

$$H_{full} = \sum_{\mathbf{q}} \int_0^{\infty} \; d\omega \; \omega \; W(\mathbf{q},\omega) \; d^{\dagger}_{\omega}(\mathbf{q}) d_{\omega}(\mathbf{q}) \tag{62}$$

Now these operators, the dressed sea-bosons in the new language obey a different kind of commutation rule:

$$[d_{\omega}(\mathbf{q}), d_{\omega'}(\mathbf{q}^{'})] = 0 \tag{63}$$

$$[d_{\omega}(\mathbf{q}), d^{\dagger}_{\omega'}(\mathbf{q}^{'})] = \delta_{\omega,\omega'}\delta_{\mathbf{q},\mathbf{q}'} \tag{64}$$

where $\delta_{\omega,\omega'}$ is a Kronecker delta function. Also the object $W(\mathbf{q},\omega)$ is a weight suitably chosen so that the the sum reduces to just the one over the collective mode in the small $\mathbf{q}$ limit. It is also clear that $W(\mathbf{q},\omega)$ has to have dimensions of inverse energy. Let us therefore postulate a form,

$$W(\mathbf{q},\omega) = -c_0\ Im(\frac{1}{\epsilon(\mathbf{q},\omega)}) \tag{65}$$

where c_0 is a suitable constant. We know from textbooks[6] that(in 3D),

$$Limit_{\mathbf{q}\rightarrow 0}W(\mathbf{q},\omega) = -c_0\ Limit_{\mathbf{q}\rightarrow 0}\ Im(\frac{1}{\epsilon(\mathbf{q},\omega)}) = \frac{c_0\pi\omega_p}{2}\delta(\omega-\omega_p) \tag{66}$$

In order that we reproduce the right collective mode we must choose,

$$c_0 = \frac{2}{\pi\omega_p} \tag{67}$$

The choices for $W(\mathbf{q},\omega)$ in other dimensions are worked out in the appendix. Now we have in our hands a convenient way of labeling excited states of our system. The eigenstates of the system have energy eigenvalues given by,

$$\Omega_{\omega}(\mathbf{q}) = \omega\ W(\mathbf{q},\omega)\ \Delta\omega \tag{68}$$

The spacing $\Delta\omega$ is the smallest possible spacing between the energies, which is arbitrarily small. Thus we can see that, in most cases it does not cost a finite energy to excite the system. This corresponds to the particle-hole mode. On the other hand, if the weight $W(\mathbf{q},\omega)$ is singular as it happens in the vicinity of $\omega = \omega_p$and $\mathbf{q} = 0$, we have a finite gap.

$$\Omega_{\omega\approx\omega_p}(\mathbf{q}\approx 0) = \omega_p\ \delta(\omega-\omega_p)\ \Delta\omega \tag{69}$$

Now if $Lim_{\omega\rightarrow\omega_p}\delta(\omega-\omega_p)\ \Delta\omega = 1$ (this defines $\Delta\omega$ if you like), then we see the emergence of the collective mode. In gapped systems we expect only the collective mode and no particle hole mode. This is possible only if for any $\omega \neq 0$,

$$W(\mathbf{q},\omega) = \infty \tag{70}$$

For the homogeneous Fermi system, this situation corresponds to Wigner crystallisation. In a Wigner crystal, in order to create excited states, you need to create phonons which require a nonzero-amount of energy each time.

Thus, all energies are allowed but each comes with a "weight" corresponding to how strong the structure factor is at that energy. It may be seen that in 3-dimensions this would mean that for small $\mathbf{q}$ the sum over i is just the collective mode, but for larger $\mathbf{q}$, we start summing the particle hole modes as well. The sums in the objects S_A and S_B have to be reinterpreted as well. It is rather unfortunate that simple and straightforward interpretations are not possible, and one is forced to take recourse to devious means such as the one we shall now describe. Let us postulate,

$$S_A(\mathbf{k}) = \sum_{\mathbf{q}}\int_0^{\infty} d\omega\ \tilde{W}(\mathbf{q},\omega)\ \frac{\bar{n}_{\mathbf{k+q}}}{(\omega-\mathbf{k}.\mathbf{q}/m-\epsilon_{\mathbf{q}})^2}g^2_{\omega}(\mathbf{q}) \tag{71}$$

$$S_B(\mathbf{k}) = \sum_{\mathbf{q}}\int_0^{\infty} d\omega\ \tilde{W}(\mathbf{q},\omega)\ \frac{1-\bar{n}_{\mathbf{k-q}}}{(\omega-\mathbf{k}.\mathbf{q}/m+\epsilon_{\mathbf{q}})^2}g^2_{\omega}(\mathbf{q}) \tag{72}$$

$$g^2_{\omega}(\mathbf{q}) = \frac{v_{\mathbf{q}}}{V}\frac{1}{\frac{\partial}{\partial\omega}\epsilon^P_{eff}(\mathbf{q},\omega)} \tag{73}$$

and,

$$\tilde{W}(\mathbf{q},\omega) \ = W(\mathbf{q},\omega)/\int_{0}^{\infty} \ d\omega \ W(\mathbf{q},\omega) \tag{74}$$

This seemingly adhoc ansatz becomes more plausible when one realises that in the small $\mathbf{q}$ limit it has a familiar form corresponding to the collective mode.

$$S_A(\mathbf{k}) \approx \sum_{\mathbf{q}=small} \frac{\bar{n}_{\mathbf{k}+\mathbf{q}}}{(\omega_c(\mathbf{q}) - \mathbf{k}.\mathbf{q}/m - \epsilon_{\mathbf{q}})^2} g_c^2(\mathbf{q}) \tag{75}$$

$$S_B(\mathbf{k}) \approx \sum_{\mathbf{q}=small} \frac{1-\bar{n}_{\mathbf{k}-\mathbf{q}}}{(\omega_c(\mathbf{q}) - \mathbf{k}.\mathbf{q}/m + \epsilon_{\mathbf{q}})^2} g_c^2(\mathbf{q}) \tag{76}$$

Let us now move on to the full propagator.

B. The Full Propagator

Let us borrow from our earlier work[4] and write down the final formulas,

$$\langle \psi^{\dagger}(\mathbf{x},t)\psi(\mathbf{x}^{'},t^{'})\rangle = |\mathcal{R}_0|^2|\mathcal{Z}_0|^4 e^{\sum_{\mathbf{k},\mathbf{q},i} U^{*i}_{\mathbf{k},\mathbf{q}}(\mathbf{x})U^{i}_{\mathbf{k},\mathbf{q}}(\mathbf{x}^{'})e^{i \ \tilde{\omega}_i(\mathbf{q})(t^{'}-t)}} e^{-\sum_{\mathbf{k},\mathbf{q}} g^{*}_{\mathbf{k},\mathbf{q}}(\mathbf{x})g_{\mathbf{k},\mathbf{q}}(\mathbf{x}^{'})e^{i \ \omega_{\mathbf{k}}(\mathbf{q})(t^{'}-t)}} \langle \psi^{\dagger}(\mathbf{x},t)\psi(\mathbf{x}^{'},t^{'})\rangle_{free} \tag{77}$$

$$\langle \psi(\mathbf{x}^{'},t^{'})\psi^{\dagger}(\mathbf{x},t)\rangle = |\mathcal{R}_0|^2|\mathcal{Z}_0|^4 e^{\sum_{\mathbf{k},\mathbf{q},i} U^{*i}_{\mathbf{k},\mathbf{q}}(\mathbf{x}^{'})U^{i}_{\mathbf{k},\mathbf{q}}(\mathbf{x})e^{i \ \tilde{\omega}_i(\mathbf{q})(t-t^{'})}} e^{-\sum_{\mathbf{k},\mathbf{q}} f^{*}_{\mathbf{k},\mathbf{q}}(\mathbf{x})f_{\mathbf{k},\mathbf{q}}(\mathbf{x}^{'})e^{i \ \omega_{\mathbf{k}}(\mathbf{q})(t-t^{'})}} \langle \psi(\mathbf{x}^{'},t^{'})\psi^{\dagger}(\mathbf{x},t)\rangle_{free} \tag{78}$$

In the above formula, the index i runs over both the collective mode as well as the particle-hole modes($i = c, \mathbf{k}_i$). As we have pointed out earlier, this must be reinterpreted to mean a weighted sum with the dynamical structure factor suitably normalised appearing as the weight.

$$g_{\mathbf{k},\mathbf{q}}(\mathbf{x}) = -e^{-i \ \mathbf{q}.\mathbf{x}}(\frac{1}{2 \ N \ \epsilon_{\mathbf{q}}})\Lambda^0_{\mathbf{k}}(-\mathbf{q})\omega_{\mathbf{k}}(\mathbf{q}) + i \ U_{\mathbf{q}}(\mathbf{x})\Lambda^0_{\mathbf{k}}(-\mathbf{q}) = -f^{*}_{\mathbf{k},\mathbf{q}}(\mathbf{x}) \tag{79}$$

Set $U_{\mathbf{q}}(\mathbf{x}) = e^{-i \ \mathbf{q}.\mathbf{x}}U_0(\mathbf{q})$

$$U_0(\mathbf{q}) = \frac{1}{N}(\frac{\theta(k_f - |\mathbf{q}|) - w_1(\mathbf{q})}{w_2(\mathbf{q})})^{\frac{1}{2}} \tag{80}$$

$$w_1(\mathbf{q}) = (\frac{1}{4 \ N \ \epsilon^2_{\mathbf{q}}})\sum_{\mathbf{k}}(\frac{\mathbf{k}.\mathbf{q}}{m})^2(\Lambda^0_{\mathbf{k}}(-\mathbf{q}))^2 \tag{81}$$

$$w_2(\mathbf{q}) = (\frac{1}{N})\sum_{\mathbf{k}}(\Lambda^0_{\mathbf{k}}(-\mathbf{q}))^2 \tag{82}$$

Here $\Lambda^0_{\mathbf{k}}(-\mathbf{q})$ is the Λ with a noninteracting momentum distribution. The interacting coefficients are given as,

$$U^{i}_{\mathbf{k},\mathbf{q}}(\mathbf{x}) = f^{*}_{\mathbf{k},\mathbf{q}}(\mathbf{x})[a_{\mathbf{k}}(\mathbf{q}), b^{\dagger}_i(\mathbf{q})] + f_{\mathbf{k},-\mathbf{q}}(\mathbf{x})[a_{\mathbf{k}}(-\mathbf{q}), b_i(\mathbf{q})] \tag{83}$$

$$\mathcal{R}_0 = exp(-\sum_{\mathbf{k},\mathbf{q},i} f^{*}_{\mathbf{k},\mathbf{q}}(\mathbf{x})f_{\mathbf{k},\mathbf{q}}(\mathbf{x})[b_i(\mathbf{q}), a^{\dagger}_{\mathbf{k}}(\mathbf{q})][a_{\mathbf{k}}(\mathbf{q}), b^{\dagger}_i(\mathbf{q})])$$

$$\times exp(-\frac{1}{2}\sum_{\mathbf{k},\mathbf{q},i} f^{*}_{\mathbf{k},\mathbf{q}}(\mathbf{x}) f^{*}_{\mathbf{k},-\mathbf{q}}(\mathbf{x})[a_{\mathbf{k}}(-\mathbf{q}), b_i(\mathbf{q})][a_{\mathbf{k}}(\mathbf{q}), b^{\dagger}_i(\mathbf{q})])$$

$$\times\ exp(-\frac{1}{2}\sum_{\mathbf{k},\mathbf{q},i} f_{\mathbf{k},\mathbf{q}}(\mathbf{x}) f_{\mathbf{k},-\mathbf{q}}(\mathbf{x})[a_{\mathbf{k}}(-\mathbf{q}), b_i(\mathbf{q})][a_{\mathbf{k}}(\mathbf{q}), b^{\dagger}_i(\mathbf{q})]) \tag{84}$$

$$\mathcal{Z}_0 = e^{i\ \sum_{\mathbf{k},\mathbf{q}\neq 0} U_0(\mathbf{q})(\frac{1}{2\ N\epsilon_{\mathbf{q}}})(\Lambda^0_{\mathbf{k}}(-\mathbf{q}))^2 \omega_{\mathbf{k}}(\mathbf{q})} e^{\frac{1}{2}\ \sum_{\mathbf{k},\mathbf{q}\neq 0}(\frac{1}{2\ N\epsilon_{\mathbf{q}}})^2(\Lambda^0_{\mathbf{k}}(-\mathbf{q}))^2(\omega_{\mathbf{k}}(\mathbf{q}))^2} e^{\frac{1}{2}\ \sum_{\mathbf{k},\mathbf{q}\neq 0}(U_0(\mathbf{q}))^2(\Lambda^0_{\mathbf{k}}(-\mathbf{q}))^2} \tag{85}$$

The commutators are given as before,

$$[b_i(\mathbf{q}), a^{\dagger}_{\mathbf{k}}(\mathbf{q})] = (\frac{\Lambda_{\mathbf{k}}(-\mathbf{q})}{\tilde{\omega}_i(\mathbf{q}) - \frac{\mathbf{k.q}}{m}})g_i(\mathbf{q}) = [a_{\mathbf{k}}(\mathbf{q}), b^{\dagger}_i(\mathbf{q})] \tag{86}$$

$$[b_i(\mathbf{q}), a_{\mathbf{k}}(-\mathbf{q})] = -(\frac{\Lambda_{\mathbf{k}}(\mathbf{q})}{\tilde{\omega}_i(\mathbf{q}) - \frac{\mathbf{k.q}}{m}})g_i(\mathbf{q}) \tag{87}$$

$$[a_{\mathbf{k}}(\mathbf{q}), b_i(-\mathbf{q})] = (\frac{\Lambda_{\mathbf{k}}(-\mathbf{q})}{\tilde{\omega}_i(-\mathbf{q}) + \frac{\mathbf{k.q}}{m}})g_i(-\mathbf{q}) \tag{88}$$

$$g^2_i(\mathbf{q}) = \frac{v_{\mathbf{q}}}{V}\frac{1}{\frac{\partial}{\partial\omega}|_{\omega=\omega_i}\epsilon^{P}_{eff}(\mathbf{q},\omega)} \tag{89}$$

C. Exchange Effects within RPA

It may be possible to employ a myriad of other similar approaches each differing from the other in how the notion of RPA is implemented. To give an example, let us rewrite the full hamiltonian differently.

$$H = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}} c_{\mathbf{k}} - \sum_{\mathbf{q}\neq 0}\frac{v(\mathbf{q})}{2V}\sum_{\mathbf{k}} n_{\mathbf{k}+\mathbf{q}/2} n_{\mathbf{k}-\mathbf{q}/2}$$

$$+\sum_{\mathbf{q}\neq 0}\frac{v(\mathbf{q})}{2V}\sum_{\mathbf{k}\neq\mathbf{k}'} n_{\mathbf{q}}(\mathbf{k}) n_{-\mathbf{q}}(\mathbf{k}^{'}) \tag{90}$$

Here the exchange term has been singled out for special treatment. We could rewrite the last term differently,

$$H = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}} c_{\mathbf{k}} - \sum_{\mathbf{q}\neq 0}\frac{v(\mathbf{q})}{2V}\sum_{\mathbf{k}} n_{\mathbf{k}+\mathbf{q}/2} n_{\mathbf{k}-\mathbf{q}/2}$$

$$-\sum_{\mathbf{q}\neq 0}\frac{v(\mathbf{q})}{2V}\sum_{\mathbf{k}\neq\mathbf{k}'} n_{\mathbf{k}-\mathbf{k}'}(\mathbf{k}/2 + \mathbf{k}^{'}/2 + \mathbf{q}/2) n_{\mathbf{k}'-\mathbf{k}}(\mathbf{k}/2 + \mathbf{k}^{'}/2 - \mathbf{q}/2) \tag{91}$$

This may be interpreted as an exchange term. The negative sign suggests precisely this. These two hamiltonians lead to different answers when the RPA is carried out on them. In fact, even the exchange term involving just the number operators can be treated in two different ways. One way is to retain it as it is, the other is to rewrite it as,

$$H_{ex} = \sum_{\mathbf{k}} \tilde{\epsilon}_{\mathbf{k}} c^{\dagger}_{\mathbf{k}} c_{\mathbf{k}} - \sum_{\mathbf{q} \neq 0} \frac{v(\mathbf{q})}{2V} \sum_{\mathbf{k}} \delta\ n_{\mathbf{k}+\mathbf{q}/2}\ \delta\ n_{\mathbf{k}-\mathbf{q}/2} \tag{92}$$

The effective dispersion includes the exchange energy,

$$\tilde{\epsilon}_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \sum_{\mathbf{q} \neq 0} \frac{v(\mathbf{q})}{V} \langle n_{\mathbf{k}-\mathbf{q}} \rangle \tag{93}$$

and $\delta\ n_{\mathbf{k}} = n_{\mathbf{k}} - \bar{n}_{\mathbf{k}}$ Again when the RPA is carried out on them they yield different answers. One has to examine this issue more closely. In particular the following questions spring to mind. Should we use one or the other or a combination of the two ? The answer may be given by appealing to our bosonization scheme. For more details, the reader is refered to our earlier works[4] ,[10]. Just to make this article self-contained, it is desirable to fix some notation.

$$n_0(\mathbf{k}) = c^{\dagger}_{\mathbf{k}} c_{\mathbf{k}} \tag{94}$$

the $a_{\mathbf{k}}(\mathbf{q})$ is a sea-boson corresponding to the fermions $c_{\mathbf{k}}$. The coefficients $\Lambda_{\mathbf{k}}(\mathbf{q}) = \sqrt{\bar{n}_{\mathbf{k}+\mathbf{q}/2}(1 - \bar{n}_{\mathbf{k}-\mathbf{q}/2})}$ are c-numbers for now.

$$c^{\dagger}_{\mathbf{k}+\mathbf{q}/2} c_{\mathbf{k}-\mathbf{q}/2} = \Lambda_{\mathbf{k}}(\mathbf{q}) a_{\mathbf{k}}(-\mathbf{q}) + a^{\dagger}_{\mathbf{k}}(\mathbf{q}) \Lambda_{\mathbf{k}}(-\mathbf{q}) \tag{95}$$

Let us now try and figure out how to decompose products of four fermions, (let us further assume that none of the four indices are equal to any other index)($\mathbf{q} \neq 0$ and $\mathbf{k} \neq \mathbf{k}^{'}$)

$$c^{\dagger}_{\mathbf{k}+\mathbf{q}/2} c_{\mathbf{k}-\mathbf{q}/2} c^{\dagger}_{\mathbf{k}^{'}-\mathbf{q}/2} c_{\mathbf{k}^{'}+\mathbf{q}/2} = [\Lambda_{\mathbf{k}}(\mathbf{q}) a_{\mathbf{k}}(-\mathbf{q}) + a^{\dagger}_{\mathbf{k}}(\mathbf{q}) \Lambda_{\mathbf{k}}(-\mathbf{q})][\Lambda_{\mathbf{k}^{'}}(-\mathbf{q}) a_{\mathbf{k}^{'}}(\mathbf{q}) + a^{\dagger}_{\mathbf{k}^{'}}(-\mathbf{q}) \Lambda_{\mathbf{k}^{'}}(\mathbf{q})] \tag{96}$$

the above equality is within RPA. It may be noted that a slightly different regrouping yields a totally different formula,

$$c^{\dagger}_{\mathbf{k}+\mathbf{q}/2} c_{\mathbf{k}-\mathbf{q}/2} c^{\dagger}_{\mathbf{k}^{'}-\mathbf{q}/2} c_{\mathbf{k}^{'}+\mathbf{q}/2} = -c^{\dagger}_{\mathbf{k}+\mathbf{q}/2} c_{\mathbf{k}^{'}+\mathbf{q}/2} c^{\dagger}_{\mathbf{k}^{'}-\mathbf{q}/2} c_{\mathbf{k}-\mathbf{q}/2}$$

$$= -[\Lambda_{\frac{\mathbf{k}}{2}+\frac{\mathbf{k}^{'}}{2}+\mathbf{q}/2}(\mathbf{k}-\mathbf{k}^{'}) a_{\frac{\mathbf{k}}{2}+\frac{\mathbf{k}^{'}}{2}+\frac{\mathbf{q}}{2}}(\mathbf{k}^{'}-\mathbf{k}) + a^{\dagger}_{\frac{\mathbf{k}}{2}+\frac{\mathbf{k}^{'}}{2}+\frac{\mathbf{q}}{2}}(\mathbf{k}-\mathbf{k}^{'}) \Lambda_{\frac{\mathbf{k}}{2}+\frac{\mathbf{k}^{'}}{2}+\frac{\mathbf{q}}{2}}(\mathbf{k}^{'}-\mathbf{k})]$$

$$\times [\Lambda_{\frac{\mathbf{k}}{2}+\frac{\mathbf{k}^{'}}{2}-\frac{\mathbf{q}}{2}}(\mathbf{k}^{'}-\mathbf{k}) a_{\frac{\mathbf{k}}{2}+\frac{\mathbf{k}^{'}}{2}-\frac{\mathbf{q}}{2}}(\mathbf{k}-\mathbf{k}^{'}) + a^{\dagger}_{\frac{\mathbf{k}}{2}+\frac{\mathbf{k}^{'}}{2}-\frac{\mathbf{q}}{2}}(\mathbf{k}^{'}-\mathbf{k}) \Lambda_{\frac{\mathbf{k}}{2}+\frac{\mathbf{k}^{'}}{2}-\frac{\mathbf{q}}{2}}(\mathbf{k}-\mathbf{k}^{'})] \tag{97}$$

Let us now examine the commutator obtained using Fermi algebra,

$$[c^{\dagger}_{\mathbf{k}-\mathbf{q}/2} c_{\mathbf{k}+\mathbf{q}/2}, c^{\dagger}_{\mathbf{k}+\mathbf{q}/2} c_{\mathbf{k}-\mathbf{q}/2} c^{\dagger}_{\mathbf{k}^{'}-\mathbf{q}/2} c_{\mathbf{k}^{'}+\mathbf{q}/2}] = [n_0(\mathbf{k}-\mathbf{q}/2) - n_0(\mathbf{k}+\mathbf{q}/2)] c^{\dagger}_{\mathbf{k}^{'}-\mathbf{q}/2} c_{\mathbf{k}^{'}+\mathbf{q}/2}$$

$$+ [\delta_{\mathbf{k}^{'}-\mathbf{q}/2, \mathbf{k}+\mathbf{q}/2}(1 - n_0(\mathbf{k}-\mathbf{q}/2)) c^{\dagger}_{\mathbf{k}+\mathbf{q}/2} c_{\mathbf{k}^{'}+\mathbf{q}/2} + \delta_{\mathbf{k}-\mathbf{q}/2, \mathbf{k}^{'}+\mathbf{q}/2} n_0(\mathbf{k}+\mathbf{q}/2) c^{\dagger}_{\mathbf{k}^{'}-\mathbf{q}/2} c_{\mathbf{k}-\mathbf{q}/2}] \tag{98}$$

It is clear that if one uses the ansatz in Eq.(96) then one obtains only the first term in Eq.(98). In order to obtain the second term it is necessary to go beyond this and try and include some portion of Eq.(97). Let us first try to use the ansatz in Eq.(97) while completely ignoring the ansatz of Eq.(96) In this case the interaction term takes a different form,

$$H_I = -\sum_{\mathbf{q} \neq 0} \sum_{\mathbf{k} \neq \mathbf{k}^{'}} \frac{v_{\mathbf{k}-\mathbf{k}^{'}}}{2V} [\Lambda_{\mathbf{k}}(\mathbf{q}) a_{\mathbf{k}}(-\mathbf{q}) + a^{\dagger}_{\mathbf{k}}(\mathbf{q}) \Lambda_{\mathbf{k}}(-\mathbf{q})][\Lambda_{\mathbf{k}^{'}}(-\mathbf{q}) a_{\mathbf{k}^{'}}(\mathbf{q}) + a^{\dagger}_{\mathbf{k}^{'}}(-\mathbf{q}) \Lambda_{\mathbf{k}^{'}}(\mathbf{q})] \tag{99}$$

Here two important changes have occured. First, the interaction $v_{\mathbf{q}}$ is replaced by $v_{\mathbf{k}-\mathbf{k}^{'}}$ and there is a change in sign. The physical meaning of these mysterious changes will be defered until later when we have investigated the details. Let us now write down the full hamiltonian,

$$H = \sum_{\mathbf{k},\mathbf{q}} \tilde{\omega}_{\mathbf{k}}(\mathbf{q}) a^{\dagger}_{\mathbf{k}}(\mathbf{q}) a_{\mathbf{k}}(\mathbf{q}) - \sum_{\mathbf{q}\neq 0} \sum_{\mathbf{k}\neq\mathbf{k}'} \frac{v_{\mathbf{k}-\mathbf{k}'}}{2V} [\Lambda_{\mathbf{k}}(\mathbf{q}) a_{\mathbf{k}}(-\mathbf{q}) + a^{\dagger}_{\mathbf{k}}(\mathbf{q}) \Lambda_{\mathbf{k}}(-\mathbf{q})][\Lambda_{\mathbf{k}'}(-\mathbf{q}) a_{\mathbf{k}'}(\mathbf{q}) + a^{\dagger}_{\mathbf{k}'}(-\mathbf{q}) \Lambda_{\mathbf{k}'}(\mathbf{q})] \tag{100}$$

Here,

$$\tilde{\omega}_{\mathbf{k}}(\mathbf{q}) = \tilde{\epsilon}_{\mathbf{k}+\mathbf{q}/2} - \tilde{\epsilon}_{\mathbf{k}-\mathbf{q}/2} \tag{101}$$

$$\tilde{\epsilon}_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \sum_{\mathbf{q}\neq 0} \frac{v_{\mathbf{q}}}{V} \bar{n}_{\mathbf{k}-\mathbf{q}} \tag{102}$$

Let us diagonalise this via a Bogoliubov transformation. To this end one assumes that the diagonalised form has the following apperance,

$$H = \sum_{i,\mathbf{q}} \omega_i(\mathbf{q}) d^{\dagger}_i(\mathbf{q}) d_i(\mathbf{q}) \tag{103}$$

$$\omega_i(\mathbf{q}) d_i(\mathbf{q}) = \sum_{\mathbf{k},\mathbf{q}} \tilde{\omega}_{\mathbf{k}}(\mathbf{q}) [d_i(\mathbf{q}), a^{\dagger}_{\mathbf{k}}(\mathbf{q})] a_{\mathbf{k}}(\mathbf{q}) + \sum_{\mathbf{k},\mathbf{q}} \tilde{\omega}_{\mathbf{k}}(-\mathbf{q}) [d_i(\mathbf{q}), a_{\mathbf{k}}(-\mathbf{q})] a^{\dagger}_{\mathbf{k}}(-\mathbf{q})$$

$$- \sum_{\mathbf{q}\neq 0} \sum_{\mathbf{k}\neq\mathbf{k}'} \frac{v_{\mathbf{k}-\mathbf{k}'}}{V} (\Lambda_{\mathbf{k}'}(\mathbf{q}) [d_i(\mathbf{q}), a_{\mathbf{k}'}(-\mathbf{q})] + [d_i(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q})] \Lambda_{\mathbf{k}'}(-\mathbf{q}))(\Lambda_{\mathbf{k}}(-\mathbf{q}) a_{\mathbf{k}}(\mathbf{q}) + a^{\dagger}_{\mathbf{k}}(-\mathbf{q}) \Lambda_{\mathbf{k}}(\mathbf{q})) \tag{104}$$

$$\omega_i(\mathbf{q}) [d_i(\mathbf{q}), a^{\dagger}_{\mathbf{k}}(\mathbf{q})] = \tilde{\omega}_{\mathbf{k}}(\mathbf{q}) [d_i(\mathbf{q}), a^{\dagger}_{\mathbf{k}}(\mathbf{q})]$$

$$- \Lambda_{\mathbf{k}}(-\mathbf{q}) \sum_{\mathbf{k}\neq\mathbf{k}'} \frac{v_{\mathbf{k}-\mathbf{k}'}}{V} (\Lambda_{\mathbf{k}'}(\mathbf{q}) [d_i(\mathbf{q}), a_{\mathbf{k}'}(-\mathbf{q})] + [d_i(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q})] \Lambda_{\mathbf{k}'}(-\mathbf{q})) \tag{105}$$

$$\omega_i(\mathbf{q}) [d_i(\mathbf{q}), a_{\mathbf{k}}(-\mathbf{q})] = -\tilde{\omega}_{\mathbf{k}}(-\mathbf{q}) [d_i(\mathbf{q}), a_{\mathbf{k}}(-\mathbf{q})]$$

$$+ \Lambda_{\mathbf{k}}(\mathbf{q}) \sum_{\mathbf{k}\neq\mathbf{k}'} \frac{v_{\mathbf{k}-\mathbf{k}'}}{V} (\Lambda_{\mathbf{k}'}(\mathbf{q}) [d_i(\mathbf{q}), a_{\mathbf{k}'}(-\mathbf{q})] + [d_i(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q})] \Lambda_{\mathbf{k}'}(-\mathbf{q})) \tag{106}$$

$$[d_i(\mathbf{q}), a^{\dagger}_{\mathbf{k}}(\mathbf{q})] = -\frac{\Lambda_{\mathbf{k}}(-\mathbf{q})}{\omega_i(\mathbf{q}) - \tilde{\omega}_{\mathbf{k}}(\mathbf{q})} \int d\vec{r}\ e^{i\mathbf{k}.\vec{r}} v(\vec{r}) \{R_1(\mathbf{q},\vec{r}) + R_2(\mathbf{q},\vec{r})\} \tag{107}$$

$$[d_i(\mathbf{q}), a_{\mathbf{k}}(-\mathbf{q})] = \frac{\Lambda_{\mathbf{k}}(\mathbf{q})}{\omega_i(\mathbf{q}) + \tilde{\omega}_{\mathbf{k}}(-\mathbf{q})} \int d\vec{r}\ e^{i\mathbf{k}.\vec{r}} v(\vec{r}) \{R_1(\mathbf{q},\vec{r}) + R_2(\mathbf{q},\vec{r})\} \tag{108}$$

Here,

$$R_1(\mathbf{q},\vec{r}) = \frac{1}{V} \sum_{\mathbf{k}'} e^{-i\mathbf{k}'.\vec{r}} \Lambda_{\mathbf{k}'}(\mathbf{q}) [d_i(\mathbf{q}), a_{\mathbf{k}'}(-\mathbf{q})] \tag{109}$$

$$R_2(\mathbf{q},\vec{r}) = \frac{1}{V} \sum_{\mathbf{k}'} e^{-i\mathbf{k}'.\vec{r}} \Lambda_{\mathbf{k}'}(-\mathbf{q}) [d_i(\mathbf{q}), a^{\dagger}_{\mathbf{k}'}(\mathbf{q})] \tag{110}$$

$$(\omega_i(\mathbf{q}) - \tilde{\omega}_{i\nabla_{\vec{r}'}}(\mathbf{q}))R_2(\mathbf{q},\vec{r}') = -\int d\vec{r}\ F_1(\vec{r}-\vec{r}';\mathbf{q})v(\vec{r})\{R_1(\mathbf{q},\vec{r}) + R_2(\mathbf{q},\vec{r})\} \tag{111}$$

$$(\omega_i(\mathbf{q}) - \tilde{\omega}_{i\nabla_{\vec{r}'}}(\mathbf{q}))R_1(\mathbf{q},\vec{r}') = \int d\vec{r}\ F_2(\vec{r}-\vec{r}';\mathbf{q})v(\vec{r})\{R_1(\mathbf{q},\vec{r}) + R_2(\mathbf{q},\vec{r})\} \tag{112}$$

$$F_1(\vec{r}-\vec{r}';\mathbf{q}) = \frac{1}{V}\sum_{\mathbf{k}} \Lambda_{\mathbf{k}}(-\mathbf{q})e^{i\mathbf{k}.(\vec{r}-\vec{r}')} \tag{113}$$

$$F_2(\vec{r}-\vec{r}';\mathbf{q}) = \frac{1}{V}\sum_{\mathbf{k}} \Lambda_{\mathbf{k}}(\mathbf{q})e^{i\mathbf{k}.(\vec{r}-\vec{r}')} \tag{114}$$

Now define,

$$R_1(\mathbf{q},\vec{r}) + R_2(\mathbf{q},\vec{r}) = R(\mathbf{q},\vec{r}) \tag{115}$$

$$(\omega_i(\mathbf{q}) - \tilde{\omega}_{i\nabla_{\vec{r}'}}(\mathbf{q}))R(\mathbf{q},\vec{r}') = \int d^3r\ f_0(\mathbf{q};\vec{r}-\vec{r}')v(\vec{r})R(\mathbf{q},\vec{r}) \tag{116}$$

$$f_0(\mathbf{q};\vec{r}-\vec{r}') = \int \frac{d^3k}{(2\pi)^3}\ n_F(\mathbf{k})e^{i\mathbf{k}.(\vec{r}-\vec{r}')}(-2i)sin(\frac{1}{2}\mathbf{q}.(\vec{r}-\vec{r}')) \tag{117}$$

$$f_0(\mathbf{q};\vec{r}-\vec{r}') = \int_0^{k_F} \frac{4\pi k\ dk}{(2\pi)^3}\ \frac{sin(k|\vec{r}-\vec{r}'|)}{|\vec{r}-\vec{r}'|}(-2i)sin(\frac{1}{2}\mathbf{q}.(\vec{r}-\vec{r}'))$$

$$= \frac{4\pi}{(2\pi)^3}\frac{(-2i)sin(\frac{1}{2}\mathbf{q}.(\vec{r}-\vec{r}'))}{|\vec{r}-\vec{r}'|}\{-k_F\ \frac{cos(k_F|\vec{r}-\vec{r}'|)}{|\vec{r}-\vec{r}'|} + \frac{sin(k_F|\vec{r}-\vec{r}'|)}{|\vec{r}-\vec{r}'|^2}\} = i\ C_0(\mathbf{q})\mathbf{q}.\nabla_{\vec{r}}\delta^3(\vec{r}-\vec{r}') \tag{118}$$

$$(\omega_i(\mathbf{q}) - i\frac{\mathbf{q}.\nabla_{\vec{r}'}}{m})R(\mathbf{q},\vec{r}') = -i\ C_0(\mathbf{q})(\mathbf{q}.\nabla_{\vec{r}'})v(\vec{r}')R(\mathbf{q},\vec{r}') \tag{119}$$

$$[\omega_i(\mathbf{q}) - i\frac{\mathbf{q}.\nabla_{\vec{r}'}}{m} + i\ C_0(\mathbf{q})(\mathbf{q}.\nabla_{\vec{r}'}v(\vec{r}'))]R(\mathbf{q},\vec{r}') = -i\ C_0(\mathbf{q})v(\vec{r}')(\mathbf{q}.\nabla_{\vec{r}'})R(\mathbf{q},\vec{r}') \tag{120}$$

This eigenvalue problem admits many eigenvalues. Actually, if not for the constraint of the square integrability of $R(\mathbf{q},\vec{r})$, for each $\mathbf{q}$ any positive real number would be an eigenvalue.

$$[\omega_i(\mathbf{q}) - i\frac{\mathbf{q}.\nabla_{\vec{r}'}}{m} + i\ C_0(\mathbf{q})(\mathbf{q}.\nabla_{\vec{r}'}v(\vec{r}'))]R(\mathbf{q},\vec{r}') = -i\ C_0(\mathbf{q})v(\vec{r}')(\mathbf{q}.\nabla_{\vec{r}'})R(\mathbf{q},\vec{r}') \tag{121}$$

Let us now solve this equation,

$$R(\mathbf{q},\vec{r}) = R(\mathbf{q},\vec{r}_0)exp(\frac{1}{i\ q\ cos\theta}\int_{r_0}^{r} dr'\ \frac{\omega_i(\mathbf{q}) + i\ C_0(\mathbf{q})\ q\ cos\theta\ \frac{\partial v(r')}{\partial r'}}{\frac{1}{m} - C_0(\mathbf{q})v(r')}) \tag{122}$$

For some suitable $\vec{r}_0$.

This work was supported by the Dept. of Physics at University of Illinois at Urbana-Champaign. The authors may be contacted at the e-mail address setlur@mrlxpa.mrl.uiuc.edu.

V. APPENDIX

In this appendix, we show how to derive the full dielectric function using the generalised RPA. Along the way we point out some pitfalls and possible generalisations. Let us write the generalised RPA hamiltonian and try and compute the dielectric function.

$$H_0 = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} n_0(\mathbf{k}) - \sum_{\mathbf{q}\neq 0} \frac{v_{\mathbf{q}}}{2V} \sum_{\mathbf{k}} n_0(\mathbf{k}+\mathbf{q}/2) n_0(\mathbf{k}-\mathbf{q}/2) \tag{123}$$

$$H_{ext}(t) = \sum_{\mathbf{q}} (U_{ext}(\mathbf{q}t) + U^{*}_{ext}(-\mathbf{q}t)) \sum_{\mathbf{k}} n_{\mathbf{q}}(\mathbf{k}) \tag{124}$$

here $n_{\mathbf{q}}(\mathbf{k}) = c^{\dagger}_{\mathbf{k}+\mathbf{q}/2} c_{bfk-\mathbf{q}/2}$ and $n_0(\mathbf{k}) = c^{\dagger}_{\mathbf{k}} c_{\mathbf{k}}$ Let us now write down the quation of motion for $n_{\mathbf{q}}(\mathbf{k})$.

$$i\frac{\partial}{\partial t} n_{\mathbf{q}}(\mathbf{k}) = \sum_{\mathbf{k}'} \epsilon_{\mathbf{k}'} [n_{\mathbf{q}}(\mathbf{k}), n_0(\mathbf{k}^{'})]$$

$$- \sum_{\mathbf{q}'\neq 0} \frac{v_{\mathbf{q}'}}{V} \sum_{\mathbf{k}'} [n_{\mathbf{q}}(\mathbf{k}), n_0(\mathbf{k}^{'})] n_0(\mathbf{k}^{'} - \mathbf{q}^{'})$$

$$+ \sum_{\mathbf{k}',\mathbf{q}'} (U_{ext}(\mathbf{q}^{'}t) + U^{*}_{ext}(-\mathbf{q}^{'}t)) [n_{\mathbf{q}}(\mathbf{k}), n_{\mathbf{q}'}(\mathbf{k}^{'})] \tag{125}$$

Now,

$$[n_{\mathbf{q}}(\mathbf{k}), n_{\mathbf{q}'}(\mathbf{k}^{'})] = [c^{\dagger}_{\mathbf{k}+\mathbf{q}/2} c_{\mathbf{k}-\mathbf{q}/2}, c^{\dagger}_{\mathbf{k}'+\mathbf{q}'/2} c_{\mathbf{k}'-\mathbf{q}'/2}]$$

$$= c^{\dagger}_{\mathbf{k}+\mathbf{q}/2} c_{\mathbf{k}'-\mathbf{q}'/2} \delta_{\mathbf{k}-\mathbf{q}/2,\mathbf{k}'+\mathbf{q}'/2} - c^{\dagger}_{\mathbf{k}'+\mathbf{q}'/2} c_{\mathbf{k}-\mathbf{q}/2} \delta_{\mathbf{k}+\mathbf{q}/2,\mathbf{k}'-\mathbf{q}'/2} \tag{126}$$

One may approximate this as,

$$[n_{\mathbf{q}}(\mathbf{k}), n_{\mathbf{q}'}(\mathbf{k}^{'})] = [c^{\dagger}_{\mathbf{k}+\mathbf{q}/2} c_{\mathbf{k}-\mathbf{q}/2}, c^{\dagger}_{\mathbf{k}'+\mathbf{q}'/2} c_{\mathbf{k}'-\mathbf{q}'/2}]$$

$$= [n_0(\mathbf{k}+\mathbf{q}/2) - n_0(\mathbf{k}-\mathbf{q}/2)] \delta_{\mathbf{k},\mathbf{k}'} \delta_{\mathbf{q},-\mathbf{q}'} \tag{127}$$

The next approximation would be to replace the number operator by its c-number expectation value. But we shall desist from that for the moment.

$$[n_{\mathbf{q}}(\mathbf{k}), n_0(\mathbf{k}^{'})] = n_{\mathbf{q}}(\mathbf{k}) (\delta_{\mathbf{k}',\mathbf{k}-\mathbf{q}/2} - \delta_{\mathbf{k}',\mathbf{k}+\mathbf{q}/2}) \tag{128}$$

$$i\frac{\partial}{\partial t} n_{\mathbf{q}}(\mathbf{k}) = (\epsilon_{\mathbf{k}-\mathbf{q}/2} - \epsilon_{\mathbf{k}+\mathbf{q}/2}) n_{\mathbf{q}}(\mathbf{k}) - \sum_{\mathbf{q}'\neq 0} \frac{v_{\mathbf{q}'}}{V} (n_0(\mathbf{k}-\mathbf{q}/2-\mathbf{q}^{'}) - n_0(\mathbf{k}+\mathbf{q}/2-\mathbf{q}^{'})) n_{\mathbf{q}}(\mathbf{k})$$

$$+ (U_{ext}(-\mathbf{q}t) + U^{*}_{ext}(\mathbf{q}t)) (n_0(\mathbf{k}+\mathbf{q}/2) - n_0(\mathbf{k}-\mathbf{q}/2)) \tag{129}$$

$$\langle n_{\mathbf{q}}(\mathbf{k}) \rangle = U_{ext}(-\mathbf{q}t) C_{\mathbf{q}}(\mathbf{k}) + U^{*}_{ext}(\mathbf{q}t) D_{\mathbf{q}}(\mathbf{k}) \tag{130}$$

Let us now ignore fluctuations in the momentum distribution. That is, we are allowed to replace

$$\langle n_0(\mathbf{k}^{'})n_{\mathbf{q}}(\mathbf{k})\rangle = \langle n_0(\mathbf{k}^{'})\rangle\langle n_{\mathbf{q}}(\mathbf{k})\rangle \tag{131}$$

$$\omega\ U_{ext}(-\mathbf{q}t)C_{\mathbf{q}}(\mathbf{k}) - \omega\ U^{*}_{ext}(\mathbf{q}t)D_{\mathbf{q}}(\mathbf{k}) = (\epsilon_{\mathbf{k}-\mathbf{q}/2} - \epsilon_{\mathbf{k}+\mathbf{q}/2})U_{ext}(-\mathbf{q}t)C_{\mathbf{q}}(\mathbf{k}) + (\epsilon_{\mathbf{k}-\mathbf{q}/2} - \epsilon_{\mathbf{k}+\mathbf{q}/2})U^{*}_{ext}(\mathbf{q}t)D_{\mathbf{q}}(\mathbf{k})$$

$$-\sum_{\mathbf{q}'\neq 0}\frac{v_{\mathbf{q}'}}{V}(\langle n_0(\mathbf{k}-\mathbf{q}/2-\mathbf{q}^{'})\rangle - \langle n_0(\mathbf{k}+\mathbf{q}/2-\mathbf{q}^{'})\rangle)U_{ext}(-\mathbf{q}t)C_{\mathbf{q}}(\mathbf{k})$$

$$-\sum_{\mathbf{q}'\neq 0}\frac{v_{\mathbf{q}'}}{V}(\langle n_0(\mathbf{k}-\mathbf{q}/2-\mathbf{q}^{'})\rangle - \langle n_0(\mathbf{k}+\mathbf{q}/2-\mathbf{q}^{'})\rangle)U^{*}_{ext}(\mathbf{q}t)D_{\mathbf{q}}(\mathbf{k})$$

$$+(U_{ext}(-\mathbf{q}t) + U^{*}_{ext}(\mathbf{q}t))(\langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle - \langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle) \tag{132}$$

$$\omega\ C_{\mathbf{q}}(\mathbf{k}) = (\epsilon_{\mathbf{k}-\mathbf{q}/2} - \epsilon_{\mathbf{k}+\mathbf{q}/2})C_{\mathbf{q}}(\mathbf{k}) - \sum_{\mathbf{q}'\neq 0}\frac{v_{\mathbf{q}'}}{V}(\langle n_0(\mathbf{k}-\mathbf{q}/2-\mathbf{q}^{'})\rangle - \langle n_0(\mathbf{k}+\mathbf{q}/2-\mathbf{q}^{'})\rangle)C_{\mathbf{q}}(\mathbf{k})$$

$$+(\langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle - \langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle) \tag{133}$$

Since,

$$U_{eff}(\mathbf{q}t) = U_{ext}(\mathbf{q}t) + \frac{v_{\mathbf{q}}}{V}\langle\rho^{'}_{-\mathbf{q}}\rangle U_{ext}(\mathbf{q}t) \tag{134}$$

$$\langle\rho^{'}_{-\mathbf{q}}\rangle = \sum_{\mathbf{k}} C_{-\mathbf{q}}(\mathbf{k}) \tag{135}$$

$$\sum_{\mathbf{k}} C_{-\mathbf{q}}(\mathbf{k}) = \sum_{\mathbf{k}}\frac{\langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle - \langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle}{\omega - \tilde{\epsilon}_{\mathbf{k}+\mathbf{q}/2} + \tilde{\epsilon}_{\mathbf{k}-\mathbf{q}/2}} \tag{136}$$

If we use the definition of the dielectric function we get a wrong answer.

$$\epsilon_{WRONG}(\mathbf{q},\omega) = \frac{U_{ext}(\mathbf{q}t)}{U_{eff}(\mathbf{q}t)} = \frac{1}{1+\frac{v_{\mathbf{q}}}{V}\sum_{\mathbf{k}}\frac{\langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle - \langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle}{\omega - \tilde{\epsilon}_{\mathbf{k}+\mathbf{q}/2} + \tilde{\epsilon}_{\mathbf{k}-\mathbf{q}/2}}} \tag{137}$$

The reason is because we have been very cavalier in our treatment of correlations. One has to consider the full Hamiltonian when dealing with the dielectric function rather than just H_0. Let us now try and do this.

$$i\frac{\partial}{\partial t}n^{t}_{\mathbf{q}}(\mathbf{k}) = (\epsilon_{\mathbf{k}-\mathbf{q}/2} - \epsilon_{\mathbf{k}+\mathbf{q}/2})n^{t}_{\mathbf{q}}(\mathbf{k}) + \sum_{\mathbf{q}'\neq 0}\frac{v_{\mathbf{q}'}}{2V}[n_{\mathbf{q}}(\mathbf{k}),\rho_{\mathbf{q}'}]\rho^{t}_{-\mathbf{q}'} + \sum_{\mathbf{q}'\neq 0}\frac{v_{\mathbf{q}'}}{2V}\rho^{t}_{\mathbf{q}'}[n_{\mathbf{q}}(\mathbf{k}),\rho_{-\mathbf{q}'}]$$

$$+\sum_{\mathbf{q}'\neq 0}(U_{ext}(\mathbf{q}^{'}t) + U^{*}_{ext}(-\mathbf{q}^{'}t))[n_{\mathbf{q}}(\mathbf{k}),\rho_{\mathbf{q}'}] \tag{138}$$

$$[n_{\mathbf{q}}(\mathbf{k}),\rho_{\mathbf{q}'}] = \sum_{\mathbf{k}'}[c^{\dagger}_{\mathbf{k}+\mathbf{q}/2}c_{\mathbf{k}-\mathbf{q}/2}, c^{\dagger}_{\mathbf{k}'+\mathbf{q}'/2}c_{\mathbf{k}'-\mathbf{q}'/2}]$$

$$= \sum_{\mathbf{k}'}c^{\dagger}_{\mathbf{k}+\mathbf{q}/2}c_{\mathbf{k}'-\mathbf{q}'/2}\delta_{\mathbf{k}-\mathbf{q}/2,\mathbf{k}'+\mathbf{q}'/2} - \sum_{\mathbf{k}'}c^{\dagger}_{\mathbf{k}'+\mathbf{q}'/2}c_{\mathbf{k}-\mathbf{q}/2}\delta_{\mathbf{k}+\mathbf{q}/2,\mathbf{k}'-\mathbf{q}'/2}$$

$$\approx \delta_{\mathbf{q},-\mathbf{q}'}[n_0(\mathbf{k}+\mathbf{q}/2) - n_0(\mathbf{k}-\mathbf{q}/2)] \tag{139}$$

$$i\frac{\partial}{\partial t}n^t_{\mathbf{q}}(\mathbf{k}) = (\epsilon_{\mathbf{k}-\mathbf{q}/2} - \epsilon_{\mathbf{k}+\mathbf{q}/2})n^t_{\mathbf{q}}(\mathbf{k}) + \frac{v_{\mathbf{q}}}{V}(n_0(\mathbf{k}+\mathbf{q}/2) - n_0(\mathbf{k}-\mathbf{q}/2))\rho^t_{\mathbf{q}}$$

$$+ (U_{ext}(-\mathbf{q}t) + U^*_{ext}(\mathbf{q}t))(n_0(\mathbf{k}+\mathbf{q}/2) - n_0(\mathbf{k}-\mathbf{q}/2)) \tag{140}$$

Let us make a first pass at the computation of the dielectric function. Here, we make use of mean-field theory, that is, replace

$$\langle n_0(\mathbf{k}^{'})\rho_{\mathbf{q}}\rangle = \langle n_0(\mathbf{k}^{'})\rangle\langle\rho_{\mathbf{q}}\rangle \tag{141}$$

$$\omega\ \langle n^t_{\mathbf{q}}(\mathbf{k})\rangle = (\epsilon_{\mathbf{k}-\mathbf{q}/2} - \epsilon_{\mathbf{k}+\mathbf{q}/2})\langle n^t_{\mathbf{q}}(\mathbf{k})\rangle + \frac{v_{\mathbf{q}}}{V}(\langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle - \langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle)\langle\rho^t_{\mathbf{q}}\rangle$$

$$+ (U_{ext}(-\mathbf{q}t) + U^*_{ext}(\mathbf{q}t))(\langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle - \langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle) \tag{142}$$

$$\langle n^t_{\mathbf{q}}(\mathbf{k})\rangle = \frac{v_{\mathbf{q}}}{V}\frac{\langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle - \langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle}{\omega - \epsilon_{\mathbf{k}-\mathbf{q}/2} + \epsilon_{\mathbf{k}+\mathbf{q}/2}}\langle\rho^t_{\mathbf{q}}\rangle$$

$$+ (U_{ext}(-\mathbf{q}t) + U^*_{ext}(\mathbf{q}t))\frac{\langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle - \langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle}{\omega - \epsilon_{\mathbf{k}-\mathbf{q}/2} + \epsilon_{\mathbf{k}+\mathbf{q}/2}} \tag{143}$$

This means,

$$\langle\rho_{-\mathbf{q}}\rangle = U_{ext}(\mathbf{q}t)\frac{P_0(\mathbf{q},\omega)}{\epsilon(\mathbf{q},\omega)} \tag{144}$$

$$P_0(\mathbf{q},\omega) = \sum_{\mathbf{k}}\frac{\langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle - \langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle}{\omega - \epsilon_{\mathbf{k}+\mathbf{q}/2} + \epsilon_{\mathbf{k}-\mathbf{q}/2}} \tag{145}$$

$$\epsilon(\mathbf{q},\omega) = 1 - \frac{v_{\mathbf{q}}}{V}P_0(\mathbf{q},\omega) \tag{146}$$

From this and the fact that

$$\epsilon_{g-RPA}(\mathbf{q},\omega) = \frac{U_{ext}(\mathbf{q}t)}{U_{eff}(\mathbf{q}t)} = \epsilon(\mathbf{q},\omega) \tag{147}$$

Next we would like to include fluctuations. Let us do this differently this time via the use of the BBGKY heirarchy.

$$i\frac{\partial}{\partial t}\langle n^t_{\mathbf{q}}(\mathbf{k})\rangle = (\epsilon_{\mathbf{k}-\mathbf{q}/2} - \epsilon_{\mathbf{k}+\mathbf{q}/2})\langle n^t_{\mathbf{q}}(\mathbf{k})\rangle + \frac{v_{\mathbf{q}}}{V}(\langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle - \langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle)\langle\rho^t_{\mathbf{q}}\rangle$$

$$+\frac{v_{\mathbf{q}}}{V}(F_{2A}(\mathbf{k}+\mathbf{q}/2,\mathbf{q}) - F_{2A}(\mathbf{k}-\mathbf{q}/2,\mathbf{q}))$$

$$+ (U_{ext}(-\mathbf{q}t) + U^*_{ext}(\mathbf{q}t))(n_0(\mathbf{k}+\mathbf{q}/2) - n_0(\mathbf{k}-\mathbf{q}/2)) \tag{148}$$

Here,

$$F_{2A}(\mathbf{k}^{'},\mathbf{q};t) = \langle n_0(\mathbf{k}^{'})\rho^t_{\mathbf{q}}\rangle - \langle n_0(\mathbf{k}^{'})\rangle\langle\rho^t_{\mathbf{q}}\rangle \tag{149}$$

$$F_2(\mathbf{k}^{'};\mathbf{k},\mathbf{q};t) = \langle n_0(\mathbf{k}^{'})n^t_{\mathbf{q}}(\mathbf{k})\rangle - \langle n_0(\mathbf{k}^{'})\rangle\langle n^t_{\mathbf{q}}(\mathbf{k})\rangle \tag{150}$$

$$F_{2A}(\mathbf{k}^{'},\mathbf{q};t) = \sum_{\mathbf{k}} F_2(\mathbf{k}^{'};\mathbf{k},\mathbf{q};t) \tag{151}$$

$$i\frac{\partial}{\partial t}F_2(\mathbf{k}^{'};\mathbf{k},\mathbf{q};t) = (\epsilon_{\mathbf{k}-\mathbf{q}/2} - \epsilon_{\mathbf{k}+\mathbf{q}/2})F_2(\mathbf{k}^{'};\mathbf{k},\mathbf{q};t) + \frac{v_{\mathbf{q}}}{V}(N(\mathbf{k}^{'},\mathbf{k}+\mathbf{q}/2) - N(\mathbf{k}^{'},\mathbf{k}-\mathbf{q}/2))\langle\rho^t_{\mathbf{q}}\rangle$$

$$+ \frac{v_{\mathbf{q}}}{V}(\langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle - \langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle)F_{2A}(\mathbf{k}^{'},\mathbf{q};t) + (U_{ext}(-\mathbf{q}t) + U^*_{ext}(\mathbf{q}t))(N(\mathbf{k}^{'},\mathbf{k}+\mathbf{q}/2) - N(\mathbf{k}^{'},\mathbf{k}-\mathbf{q}/2)) \tag{152}$$

Let us write,

$$F_2(\mathbf{k}^{'};\mathbf{k},\mathbf{q};t) = U_{ext}(-\mathbf{q}t)F_{2,a}(\mathbf{k}^{'};\mathbf{k},\mathbf{q}) + U^*_{ext}(\mathbf{q}t)F_{2,b}(\mathbf{k}^{'};\mathbf{k},\mathbf{q}) \tag{153}$$

$$\langle\rho^t_{\mathbf{q}}\rangle = U_{ext}(-\mathbf{q}t)\langle\rho^{'}_{\mathbf{q}}\rangle + U^*_{ext}(\mathbf{q}t)\langle\rho^{''}_{\mathbf{q}}\rangle \tag{154}$$

Also define,

$$N(\mathbf{k},\mathbf{k}^{'}) = \langle n_0(\mathbf{k})n_0(\mathbf{k}^{'})\rangle - \langle n_0(\mathbf{k})\rangle\langle n_0(\mathbf{k}^{'})\rangle \tag{155}$$

$$\omega\ F_{2,a}(\mathbf{k}^{'};\mathbf{k},\mathbf{q}) = (\epsilon_{\mathbf{k}-\mathbf{q}/2} - \epsilon_{\mathbf{k}+\mathbf{q}/2})F_{2,a}(\mathbf{k}^{'};\mathbf{k},\mathbf{q}) + \frac{v_{\mathbf{q}}}{V}(N(\mathbf{k}^{'},\mathbf{k}+\mathbf{q}/2) - N(\mathbf{k}^{'},\mathbf{k}-\mathbf{q}/2))\langle\rho^{'}_{\mathbf{q}}\rangle$$

$$+ \frac{v_{\mathbf{q}}}{V}(\langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle - \langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle)F^a_{2A}(\mathbf{k}^{'},\mathbf{q}) + (N(\mathbf{k}^{'},\mathbf{k}+\mathbf{q}/2) - N(\mathbf{k}^{'},\mathbf{k}-\mathbf{q}/2)) \tag{156}$$

$$-\omega\ F_{2,b}(\mathbf{k}^{'};\mathbf{k},\mathbf{q}) = (\epsilon_{\mathbf{k}-\mathbf{q}/2} - \epsilon_{\mathbf{k}+\mathbf{q}/2})F_{2,b}(\mathbf{k}^{'};\mathbf{k},\mathbf{q}) + \frac{v_{\mathbf{q}}}{V}(N(\mathbf{k}^{'},\mathbf{k}+\mathbf{q}/2) - N(\mathbf{k}^{'},\mathbf{k}-\mathbf{q}/2))\langle\rho^{''}_{\mathbf{q}}\rangle$$

$$+ \frac{v_{\mathbf{q}}}{V}(\langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle - \langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle)F^b_{2A}(\mathbf{k}^{'},\mathbf{q}) + (N(\mathbf{k}^{'},\mathbf{k}+\mathbf{q}/2) - N(\mathbf{k}^{'},\mathbf{k}-\mathbf{q}/2)) \tag{157}$$

$$\epsilon(\mathbf{q},\omega)\ F^a_{2A}(\mathbf{k}^{'},\mathbf{q}) = \frac{v_{\mathbf{q}}}{V}\sum_{\mathbf{k}}\frac{N(\mathbf{k}^{'},\mathbf{k}+\mathbf{q}/2) - N(\mathbf{k}^{'},\mathbf{k}-\mathbf{q}/2)}{\omega - \epsilon_{\mathbf{k}-\mathbf{q}/2} + \epsilon_{\mathbf{k}+\mathbf{q}/2}}\langle\rho^{'}_{\mathbf{q}}\rangle$$

$$+ \sum_{\mathbf{k}}\frac{N(\mathbf{k}^{'},\mathbf{k}+\mathbf{q}/2) - N(\mathbf{k}^{'},\mathbf{k}-\mathbf{q}/2)}{\omega - \epsilon_{\mathbf{k}-\mathbf{q}/2} + \epsilon_{\mathbf{k}+\mathbf{q}/2}} \tag{158}$$

$$\omega\ C_{\mathbf{q}}(\mathbf{k}) = (\epsilon_{\mathbf{k}-\mathbf{q}/2} - \epsilon_{\mathbf{k}+\mathbf{q}/2})C_{\mathbf{q}}(\mathbf{k}) + \frac{v_{\mathbf{q}}}{V}(\langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle - \langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle)\langle\rho^{'}_{\mathbf{q}}\rangle + \frac{v_{\mathbf{q}}}{V}(F^a_{2A}(\mathbf{k}+\mathbf{q}/2) - F^a_{2A}(\mathbf{k}-\mathbf{q}/2))$$

$$+ (\langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle - \langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle) \tag{159}$$

$$C_{\mathbf{q}}(\mathbf{k}) = \frac{v_{\mathbf{q}}}{V}\frac{\langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle - \langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle}{\omega - \epsilon_{\mathbf{k}-\mathbf{q}/2} + \epsilon_{\mathbf{k}+\mathbf{q}/2}}\langle\rho^{'}_{\mathbf{q}}\rangle$$

$$+\frac{\langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle - \langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle}{\omega - \epsilon_{\mathbf{k}-\mathbf{q}/2} + \epsilon_{\mathbf{k}+\mathbf{q}/2}}$$

$$+\frac{v_{\mathbf{q}}}{V}\frac{F^a_{2A}(\mathbf{k}+\mathbf{q}/2,\mathbf{q}) - F^a_{2A}(\mathbf{k}-\mathbf{q}/2,\mathbf{q})}{\omega - \epsilon_{\mathbf{k}-\mathbf{q}/2} + \epsilon_{\mathbf{k}+\mathbf{q}/2}} \tag{160}$$

After all this, it may be shown that the overall dielectric function including possible fluctutations in the momentum distribution is given by,

$$\epsilon_{eff}(\mathbf{q},\omega) = \epsilon_{g-RPA}(\mathbf{q},\omega) - (\frac{v_{\mathbf{q}}}{V})^2\frac{P_2(\mathbf{q},\omega)}{\epsilon_{g-RPA}(\mathbf{q},\omega)} \tag{161}$$

Here,

$$P_2(\mathbf{q},\omega) = \sum_{\mathbf{k},\mathbf{k}'}\frac{N(\mathbf{k}+\mathbf{q}/2,\mathbf{k}^{'}+\mathbf{q}/2) - N(\mathbf{k}-\mathbf{q}/2,\mathbf{k}^{'}+\mathbf{q}/2) - N(\mathbf{k}+\mathbf{q}/2,\mathbf{k}^{'}-\mathbf{q}/2) + N(\mathbf{k}-\mathbf{q}/2,\mathbf{k}^{'}-\mathbf{q}/2)}{(\omega - \epsilon_{\mathbf{k}-\mathbf{q}/2} + \epsilon_{\mathbf{k}+\mathbf{q}/2})(\omega - \epsilon_{\mathbf{k}'-\mathbf{q}/2} + \epsilon_{\mathbf{k}'+\mathbf{q}/2})} \tag{162}$$

$$\epsilon_{g-RPA}(\mathbf{q},\omega) = 1 + \frac{v_{\mathbf{q}}}{V}\sum_{\mathbf{k}}\frac{\langle n_0(\mathbf{k}+\mathbf{q}/2)\rangle - \langle n_0(\mathbf{k}-\mathbf{q}/2)\rangle}{\omega - \epsilon_{\mathbf{k}+\mathbf{q}/2} + \epsilon_{\mathbf{k}-\mathbf{q}/2}} \tag{163}$$

---

[1] See for example, "*Quantum Mechanics*" A.A. Sokolov, I.M. Ternov and V.Ch.Zhukovskii, Mir Publishers, Moscow, 1984

[2] D. Bohm and D. Pines, Phys. Rev. **92**, 609 (1953)

[3] Bogoliubov N. N., J. Phys. Moscow **11**, 23 (1947), Bogoliubov N. N. and D. N. Zubarev, Sov. Phys. JETP **1**, 83 (1955)

[4] G.S. Setlur and Y.C. Chang, Phys. Rev. B, **57**, no. 24, 15 144(1998)

[5] J.D. Weeks, K.Katsov, K. Vollmayr, Phys. Rev. Lett., **81**, no. 20, 4400 (1998)

[6] G.D.Mahan "*Many-Particle Physics*", II ed.,Plenum Press, New York, 1990

[7] J. Lindhard, K. Dan. Vidensk. Selsk. Mat. Fys. Medd. **28**, (8) (1954)

[8] L. Kadanoff and G. Baym "*Quantum Statistical Mechanics*" Addison Wesley(Advanced Book Classics), Redwood City, 1991

[9] G.S.Setlur, M.T. Kuo and Y.C. Chang "Corrections to RPA Induced by Number Fluctuations", APS e-print, 1998

[10] G.S.Setlur and Y.C. Chang, LANL preprint cond-mat/9810308

[11] G.S.Setlur and Y.C. Chang, LANL preprint cond-mat/9811238

[12] The pathological exception is for nonideal distributions such as, $\langle n_{\mathbf{k}}\rangle = 0$ for $|\mathbf{k}| < 0.5\ k_f$ and $\langle n_{\mathbf{k}}\rangle = 1$ for $0.5\ k_f < |\mathbf{k}| < 1.5\ k_f$ and $\langle n_{\mathbf{k}}\rangle = 0$ for $|\mathbf{k}| > 1.5\ k_f$ this is very nonideal but has zero fluctuation, but hopefully these are not physical.